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# Statics of special types of homogeneous elastic slabs with variable thickness

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# STATICS OF SPECIAL TYPES OF
# HOMOGENEOUS ELASTIC SLABS WITH VARIABLE THICKNESS

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# I. INTRODUCTION.

Within the last few years a great deal of work has been done on various types of elastic plate and shell problems. However, practically all of this study has been concerned with plates of uniform thickness and plane central or elastic surface. In a few cases circular plates of variable thickness have been considered, and some attention has been focused on shells whose surfaces are sections of co-axial cylinders. In view of the modern paving slab with its curved cross-section and thickened edges (note sketch), it seems worth while to devote some study to rectangular plates of variable thickness and curved elastic surface.

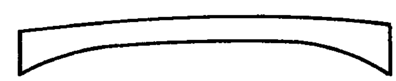


Fig. 1.

This paper is presented as a beginning in the mathematical or theoretical analysis of the statics of such slabs. The results obtained here indicate approximation methods and further developments which may lead to very useful results. The problem is complicated by the fact that, in a theoretical investigation, the elastic type of support given a paving slab by the earth upon which it rests is a source of difficulty. Even in flat slabs the treatment of this type of support has offered a serious obstacle. Hence, this paper is limited to a discussion of slabs with edges either simply supported or clamped. The treatment seems to divide itself into four distinct parts which are outlined briefly in the following paragraphs.

Chapter two deals with general shell theory. In it the development of the differential equation of deflection in terms of the slab thickness and curvature at any point is carried out. This differential equation is a partial differential equation of the fourth order, and has coefficients which are expressed in terms of the thickness and original curvature of the slab. Since only slabs of cylindrical elastic surface are to be considered, this curvature causes no difficulty in so far as obtaining the solution is concerned. However, the type of function which expresses the variation in thickness does influence the ease of obtaining the solution.

Hence, the third chapter deals with the finding of a thickness function which will fit the type of slab desired and still leave a differential equation which can be solved with a reasonable degree of ease. An exponential thickness expression is found to be one which gives constant coefficients and produces a slab of approximately the required shape.

In the next two chapters the general theory, subject to the just mentioned thickness function, is applied to several special cases. First is considered an infinite slab with a load-distribution which depends upon the lateral but not upon the longitudinal coördinate. The elastic surface of the slab is thought of as a section of an infinitely long cylinder. The deflections, moments, and shears of such a slab, subjected to a uniform load and having simple and clamped-edge supports, are worked out. These are compared with the deflections, moments, and shears in a beam of unit width and longitudinal-section similar to the cross-section of the plate.

The large radius of curvature of the simply supported plate seems to have such little effect upon the moments, shears, etc., that in the more complicated remainder of the paper this is left out of consideration completely, the elastic surface being considered plane. This last part deals with the effect of point, line, and rectangular loads upon the deflection of a simply supported rectangular plate of finite dimensions and with a variable cross-section defined by the thickness function developed in the first chapter. The deflection is obtained first by a method which H. Schmidt* has applied to the uniform plate. Then this same result is obtained by another method similar to one used by A. Nadai** in his treatment of uniform plates. By making the thickness function constant in these solutions and comparing their degenerate form with the form of solutions of uniform plates, the results of these two processes are shown to be consistent with the known solutions for uniform plates.

*Schmidt, H. Biegung der frei aufliegender Rechteckplatte mit statischer, rechteckig berandeter Lastverteilung. Zs. f. Phys., 68:423-432. 1931.

**Nadai, A. Elastische Platten. Julius Springer, Berlin. 1925.

## II. DEVELOPMENT OF THE DEFLECTION THEORY.

### 1. Definitions and Limitation of Plate Theory.

In this paper a rectangular slab with an initially cylindrical middle surface of constant radius $R$ is assumed. The origin is taken at one edge of the elastic surface of the slab, as is shown in figure 2. The X-axis is parallel to the axis of the cylinder, and the Y-axis is an arc in the cylindrical surface and in a plane perpendicular to the X-axis. The third or deflection coördinate $z$ is measured along a radial line toward the axis of the cylinder. Thus the slab is bounded by the three sets of surfaces: $x = 0$ and $x = a$, $y = 0$ and $y = L$, and an upper and lower surface defined by a single-valued, continuous thickness function $t = t(x,y)$. This function is so formed that at any point $(x,y)$ the radial distance from the elastic surface to the upper surface is the same as the radial distance from the elastic surface to the lower surface. In addition, the thickness varies so slowly from point to point that planes tangent to the upper and lower surfaces at the opposite ends of a radial line through the

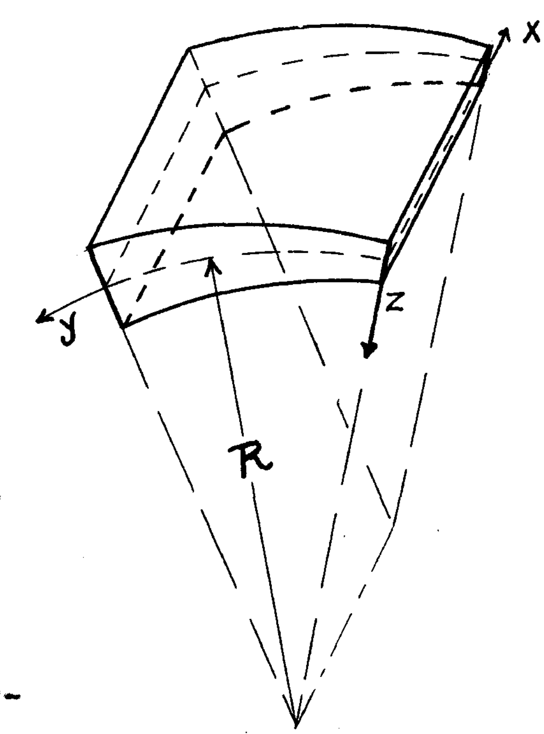



Fig. 2.

plate are nearly parallel. The radius R is very large in comparison with the plate thickness.

The development of the deflection expression depends upon an hypothesis, similar to the Bernoulli and Navier hypothesis for beams*, which assumes that the thickness of the plate is small in comparison with its other dimensions and further that the deflection of the plate is small in comparison with its thickness. This being accepted, it may be said further:

a. A radial line drawn through the slab before bending remains straight and perpendicular to the deflected elastic surface after bending.

b. There is no appreciable stretching of the middle surface. Practically all the energy imparted to the slab by the external forces is exhausted in bending.

## 2. Notation.

The symbolism adopted in this study is similar to that employed by A. Nadai** and H. M. Westergaard.*** (Refer to figure 3).

$P_1$ is a normal stress in the direction of $x$ acting upon a unit of section normal to the X-axis.

*Geckeler, J. W. Elastostatik. Handbuch der Physik, 6:211-213. Julius Springer, Berlin. 1928.

**Nadai, A. Elastische Platten. Julius Springer, Berlin. 1925.

***Westergaard, H. M. Computation of stresses in bridge slabs due to wheel loads. Pub. Roads, 11:2. 1930.

$P_2$ is a normal stress in the direction of $y$ acting upon a unit of section normal to the Y-axis.

$P_{12}$ is a tangential shearing stress in the direction of $y$ acting upon a unit of section normal to the X-axis.

$P_{21}$ is a horizontal shearing stress in the direction of $x$ acting upon a unit of section normal to the Y-axis.

$Q_1$ is the radial shear in the direction of $z$ acting upon a unit of section normal to the X-axis. It is considered positive when acting upward upon the edge of increasing value of $x$.

$Q_2$ is the radial shear in the direction of $z$ acting upon a unit of section normal to the Y-axis. It is considered positive when acting upward upon the edge of increasing values of $y$.

$M_1$ is the bending moment on a unit of section normal to the X-axis. It is considered positive when it produces compression in the upper surface.

$M_2$ is the bending moment on a unit of section normal to the Y-axis. It is considered positive when it produces compression in the upper surface.

$M_{12}$ and $M_{21}$ are torsional moments, and are indicated in figure 3. Since the shearing stresses which determine these moments act upon nearly orthogonal planes, these stresses, as well as their corresponding torsional moments, are considered equal. A twisting moment is considered positive if it tends to produce a compres-

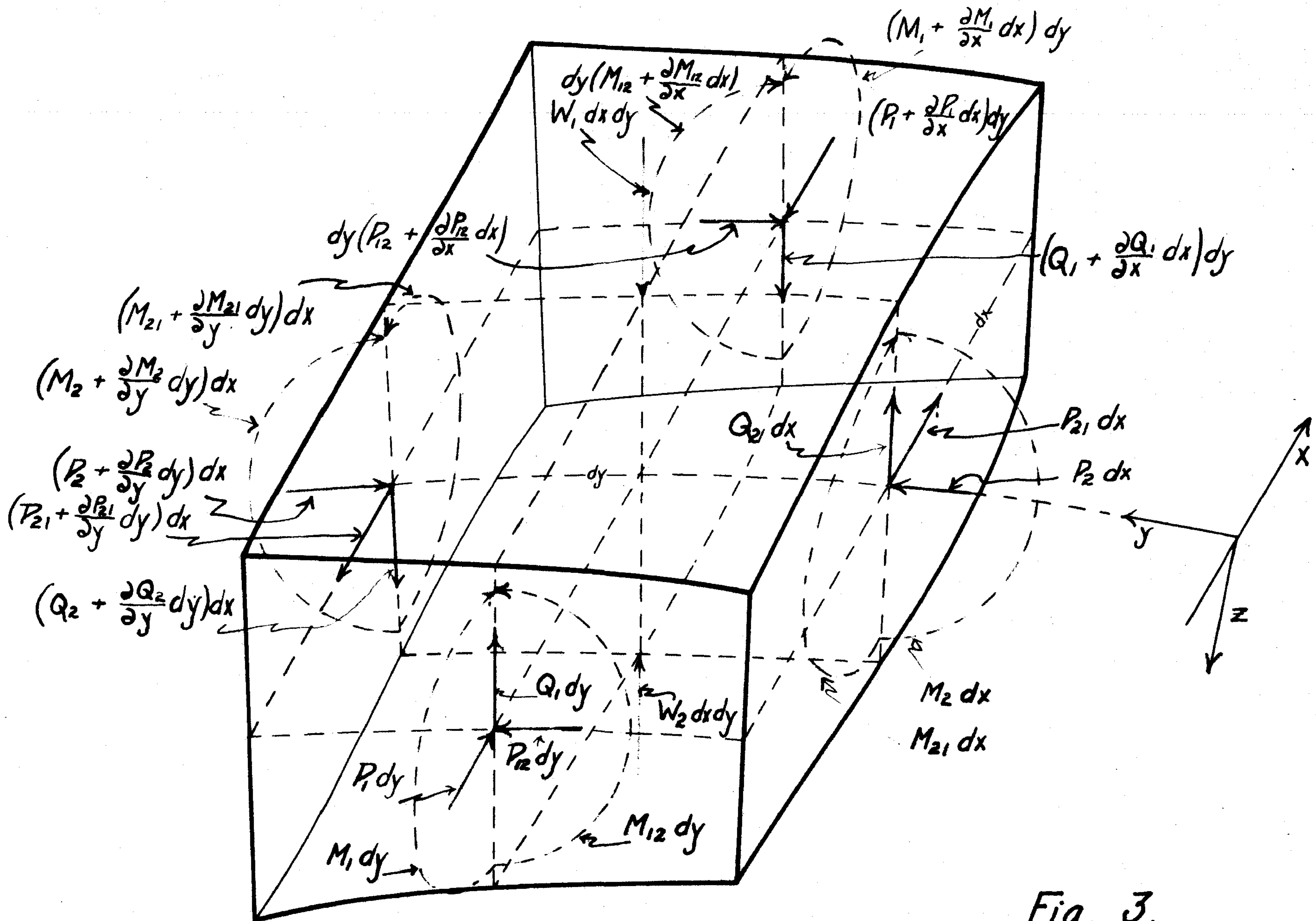

$(M_1 + \frac{\partial M_1}{\partial x} dx) dy$
$dy(M_{12} + \frac{\partial M_{12}}{\partial x} dx)$
$W_1 \, dx \, dy$
$(P_1 + \frac{\partial P_1}{\partial x} dx) dy$
$dy(P_{12} + \frac{\partial P_{12}}{\partial x} dx)$
$(Q_1 + \frac{\partial Q_1}{\partial x} dx) dy$
$(M_{21} + \frac{\partial M_{21}}{\partial y} dy) dx$
$(M_2 + \frac{\partial M_2}{\partial y} dy) dx$
$(P_2 + \frac{\partial P_2}{\partial y} dy) dx$
$(P_{21} + \frac{\partial P_{21}}{\partial y} dy) dx$
$(Q_2 + \frac{\partial Q_2}{\partial y} dy) dx$
$dx$
$dy$
$Q_{21} \, dx$
$P_{21} \, dx$
$P_2 \, dx$
$x$
$y$
$z$
$M_2 \, dx$
$M_{21} \, dx$
$Q_1 \, dy$
$W_2 \, dx \, dy$
$P_{12} \, dy$
$P_1 \, dy$
$M_{12} \, dy$
$M_1 \, dy$

Fig. 3.

sion in the upper surface in a diagonal direction of increasing $x$ and $y$.

$\sigma_1$ is the normal horizontal unit stress in the direction of $x$. It is positive when a tension .

$\sigma_2$ is the normal tangential unit stress in the direction of $y$. It is positive when a tension.

$\tau_{12}$ and $\tau_{21}$ are unit shearing stresses. The first subscript corresponds to the normal to the surface element in which they lie (1 to x, 2 to y). The second subscript shows the axis to which they are parallel. They are considered positive when they tend to produce elongation in the diagonal direction of increasing x and y.

$\varepsilon_1$ and $\varepsilon_2$ are unit elongations in the directions of $x$ and $y$ respectively.

$\gamma_{12}$ and $\gamma_{21}$ are the unit shearing strains corresponding to $\tau_{12}$ and $\tau_{21}$ respectively.

$E$ is Young's modulus.

$\mu$ is Poisson's ratio.

$t = t(x,y)$ is the single-valued, continuous function which defines the slab thickness at any point.

$$N = \frac{Et^3}{12(1-\mu^2)} .$$

$$N' = \frac{dN}{dy} . \qquad N'' = \frac{d^2N}{dy^2} . \qquad E_1 = \frac{E}{12(1-\mu^2)} .$$

$K = 2N'/N.$ $\qquad k = N''/N.$

$W_1 = W_1(x,y)$ is a unit function which shows the load on the upper surface of the slab.

$W_2 = W_2(x,y)$ is a unit function which shows the load on the lower surface of the slab.

$$\Delta = \frac{\delta^2}{\delta x^2} + \frac{\delta^2}{\delta y^2}, \qquad \Delta\Delta = \frac{\delta^4}{\delta x^4} + 2\frac{\delta^4}{\delta x^2\delta y^2} + \frac{\delta^4}{\delta y^4}.$$

## 3. Moment and Equilibrium Equations.

Figure 3 is a differential element of dimensions $t$, $dx$, $dy$ taken from the slab sketched in figure 2. It shows the stresses and moments acting upon such an element. A resolution of these forces and moments parallel to the three coördinate axes leads to six equations. In accordance with the large magnitude of $R$, these can be written in a very simple form:

$$\begin{aligned} &\frac{\delta P_1}{\delta x} + \frac{\delta P_{21}}{\delta y} = 0, \\ &\frac{\delta P_2}{\delta y} + \frac{\delta P_{12}}{\delta x} + \frac{Q_1}{R} = 0, \\ &\frac{\delta Q_2}{\delta y} + \frac{\delta Q_1}{\delta x} - \frac{P_2}{R} = W_2 - W_1. \end{aligned} \tag{2.1}$$

$$\frac{\delta M_1}{\delta x} + \frac{\delta M_{21}}{\delta y} = Q_1 ,$$

$$\frac{\delta M_2}{\delta y} + \frac{\delta M_{12}}{\delta x} = Q_2 , \qquad (2.2)$$

$$P_{12} - P_{21} = M_{21}/R .$$

If $P_{21}$, $P_{12}$, and $M_{21}$ are almost equally significant in producing the shearing stresses, since $R$ is large, the term $M_{21}/R$ may be neglected. If the central shears are significant at all, the last equation may be replaced by

$$P_{12} = P_{21} .$$

It has already been assumed that $M_{12} = M_{21}$.

The sum of the partial derivative of the first equation of (2.2) with respect to $x$ and of the second with respect to $y$, when taken in conjunction with the third equation of (2.1), leads to

$$\frac{\delta^2 M_1}{\delta x^2} + 2\frac{\delta^2 M_{12}}{\delta x \delta y} + \frac{\delta^2 M_2}{\delta y^2} = \frac{\delta Q_1}{\delta x} + \frac{\delta Q_2}{\delta y} = \frac{P_2}{R} + W_2 - W_1 . \quad (2.3)$$

## 4. Unit Stresses and Strains.

At the middle surface of the slab the three unit stresses are

$$\sigma_1 = -P_1/t, \quad \sigma_2 = -P_2/t, \quad \tau_{12} = -P_{12}/t . \qquad (2.4)$$

The slab action outlined in hypothesis a of section one indicates a linear distribution of stress in a radial direction. In consideration of this, the unit stresses at the surfaces are

$$\sigma_1 = -P_1/t \mp 6M_1/t^2, \quad \sigma_2 = -P_2/t \mp 6M_2/t^2,$$
$$\tau_{12} = -P_{12}/t \mp 6M_{12}/t^2. \tag{2.5}$$

From the theory of elasticity*

$$\varepsilon_1 = (\sigma_1 - \mu\sigma_2)/E, \quad \varepsilon_2 = (\sigma_2 - \mu\sigma_1)/E,$$
$$\gamma_{12} = 2(1 + \mu)\tau_{12}/E, \tag{2.6}$$

$$\sigma_1 = E(\varepsilon_1 + \mu\varepsilon_2)/(1 - \mu^2), \quad \sigma_2 = E(\varepsilon_2 + \mu\varepsilon_1)/(1 - \mu^2),$$
$$\tau_{12} = E\gamma_{12}/(2 + 2\mu). \tag{2.7}$$

## 5. Curvature of the Middle Surface.

In order to find the relation between the moments and the deflection of the elastic middle surface, it is necessary to note the relation of the moments to the change in curvature. As the initial curvature in the $x$ direction is zero, the final expression for the curvature is given in sufficiently accurate form by the negative of the second partial derivative of $z$ with

*Prescott, John. Applied elasticity. p. 28. Longmans, Green and Co. London. 1924.

respect to $x$. This change in curvature is connected with $M_1$ and $M_2$ in the following manner:

$$\frac{\delta^2 z}{\delta x^2} = \frac{-12(M_1 - \mu M_2)}{Et^3} .$$

In the $y$ direction, the initial curvature is $(-1/R)$. The final curvature may be resolved into two parts: the first, caused by the displacement of each element by the same amount in the radial direction; the second, by a displacement with one point fixed. (See figure 4).

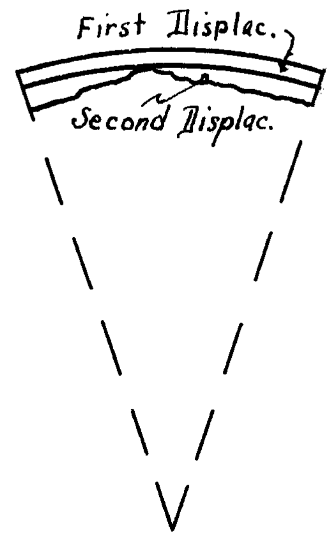



Fig. 4.

The first curvature is

$$-\frac{1}{R} + \frac{1}{R + \delta R} = \frac{-R - \delta R + R}{R^2 + R\delta R} \rightarrow \frac{\delta R}{R^2} = \frac{-z}{R^2} .$$

The second part is represented by

$$-\frac{\delta^2 z}{\delta y^2} ,$$

so that the total curvature in that direction is

$$-\frac{\delta^2 z}{\delta y^2} - \frac{z}{R^2} .$$

Hence, one gets another curvature-moment equation

$$\frac{z}{R^2} + \frac{\delta^2 z}{\delta y^2} = \frac{-12(M_2 - \mu M_1)}{Et^2} .$$

The effect of the twisting moment is expressed by the following:

$$\frac{\delta^2 z}{\delta x \delta y} = \frac{12(1 + \mu)M_{12}}{Et^3}\text{*} .$$

## 6. Moments in Terms of the Deflection of the Middle Surface.

The three equations of the last paragraph may be combined and rewritten to give an equation connecting each moment and the deflection of the elastic surface; e.g.,

$$M_1 = - N \quad \frac{\delta^2 z}{\delta x^2} + \mu \frac{\delta^2 z}{\delta y^2} + \mu \frac{z}{R^2} \quad ,$$

$$M_2 = - N \quad \frac{\delta^2 z}{\delta y^2} + \mu \frac{\delta^2 z}{\delta x^2} + \frac{z}{R^2} \quad , \qquad (2.8)$$

$$M_{12} = - N \quad 1 - \mu \quad \frac{\delta^2 z}{\delta x \delta y} \quad .$$

*Westergaard, H. M. Computation of stresses in bridge slabs due to wheel loads. Pub. Roads, 11:4. 1930.

## 7. The Deflection Equation.

When these values (2.8) for the moments are substituted in equation (2.3), a new differential equation results:

$$\Delta(N\Delta z) - (1-\mu)\left(\frac{\delta^2 N}{\delta x^2}\frac{\delta^2 z}{\delta y^2} - 2\frac{\delta^2 N}{\delta x\delta y}\frac{\delta^2 z}{\delta x\delta y} + \frac{\delta^2 N}{\delta y^2}\frac{\delta^2 z}{\delta x^2}\right)$$

$$+\frac{1}{R^2}\left[\left(\frac{\delta^2 N}{\delta y^2} + \mu\frac{\delta^2 N}{\delta x^2}\right) + 2\left(\frac{\delta N}{\delta y}\frac{\delta z}{\delta y} + \mu\frac{\delta N}{\delta x}\frac{\delta z}{\delta x}\right)\right. \tag{2.9}$$

$$\left.+ N\left(\frac{\delta^2 z}{\delta y^2} + \mu\frac{\delta^2 z}{\delta x^2}\right)\right] = W_1 - W_2 - \frac{P_2}{R}.$$

If $R \longrightarrow \infty$, this equation simplifies to

$$\Delta(N\Delta z) - (1-\mu)\left(\frac{\delta^2 N}{\delta x^2}\frac{\delta^2 z}{\delta y^2} - 2\frac{\delta^2 N}{\delta x\delta y}\frac{\delta^2 z}{\delta x\delta y} + \frac{\delta^2 N}{\delta y^2}\frac{\delta^2 N}{\delta x^2}\right) = W_1 - W_2;$$

and the latter equation corresponds to one given for this case by K. Hohenemser*.

If $t$ is a function of $y$ alone, equation (2.9) becomes

$$\Delta(\Delta z) + \frac{\mu}{R^2}\frac{\delta^2 z}{\delta x^2} + \frac{1}{R^2}\frac{\delta^2 z}{\delta y^2} + K\left(\frac{\delta^3 z}{\delta y^3} + \frac{\delta^3 z}{\delta x^2\delta y} + \frac{1}{R^2}\frac{\delta z}{\delta y}\right)$$

$$+ k\left(\frac{\delta^2 z}{\delta y^2} + \mu\frac{\delta^2 z}{\delta x^2} + \frac{z}{R^2}\right) - \frac{1}{N}\left(W_1 - W_2 - \frac{P_2}{R}\right) = 0. \tag{2.10}$$

*Hohenemser, K. Die Methoden zur angenäherten Lösung von Eigenwertenproblemen in der Elastokinetik. p. 86. Julius Springer, Berlin. 1932.

This equation checks with one developed by H. M. Westergaard* in his study of the Stevenson Creek Dam.

If $t$ is a function of $y$ alone and if $R \longrightarrow$ , as in the case of a plate with a plane central surface, equation (2.9) becomes

$$\Delta(\Delta z) + K\,\frac{\delta(\Delta z)}{\delta y} + k\left(\frac{\delta^2 z}{\delta y^2} + \mu\,\frac{\delta^2 z}{\delta x^2}\right) - \frac{1}{N}\left(W_1 - W_2\right) = 0. \quad (2.11)$$

If $t$ is a function of $y$ alone and the load is also a function of $y$ alone, equation (2.9) takes a much simpler form:

$$z^{(4)} + z''/R^2 + K(z^{(3)} + z'/R^2) + k(z'' + z/R^2) \qquad (2.12)$$
$$-(W_1 - W_2 - P_2/R)/N = 0,$$

where

$$z^{(4)} = \frac{d^4 z}{dy^4}, \quad z^{(3)} = \frac{d^3 z}{dy^3}, \quad z'' = \frac{d^2 z}{dy^2}, \quad z' = \frac{dz}{dy}.$$

This paper will be concerned with the last two cases as expressed by equations (2.11) and (2.12).

*Westergaard, H. M. Theoretical analysis of the structural action of the stevenson creek arch dam. Am. So. Civil Eng. Proceedings, 54:242. 1928.

## III. DEVELOPMENT OF A THICKNESS FUNCTION WHICH SIMPLIFIES THE DIFFERENTIAL EQUATION.

As was indicated in the introduction, the type of shell which instigated this research was one of the general shape of the present paving slab. An approximation for such a slab is found in a shell whose central surface is a cylinder of large radius (neighborhood of 100 units) and whose thickness varies in the $y$ direction from $t_1$ at the edge $y = 0$ to $t_2$ at the edge $y = L$. The thickness in the $x$ direction is independent of that coördinate. (The two edge thicknesses $t_1$ and $t_2$ are considered to be in the neighborhood of 1/2 and 3/4 units respectively; and L is thought of as being in the neighborhood of 10 units.)

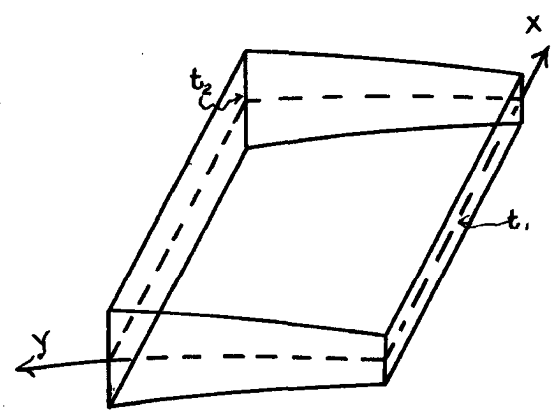



Fig. 5.

It is next necessary to find a thickness function which will satisfy these conditions and still provide a deflection equation which can be solved with a reasonable degree of ease. Choosing this function from the standpoint of the symmetry and simplicity of its form usually leads to a fourth order partial deflection equation with variable coefficients. The difficulty of solving such an equation suggests that one try to find a function which will give constant coefficients in equation (2.10) and still provide a slab with a cross-section of the general nature indicated in figure 5.

The two major coefficients of (2.10) are $K$ and $k$, where

$$K = 2N'/N = 6t'/t, \quad k = N''/N = 6(t')^2/t^2 + 3t''/t. \tag{3.1}$$

A thickness function, satisfying the necessary differentiability conditions, of the form

$$t(y) = t_1\sqrt[3]{g_1(y)}, \tag{3.2}$$

where

$$g_1 = e^{3cy}, \quad c = (\log t_2/t_1)/L>0, \tag{3.3}$$

will make both $K$ and $k$ constant, giving them the following values:

$$K = 6c, \quad k = 9c^2. \tag{3.4}$$

In addition, the constant $c$ has been determined so that $t$ becomes $t_1$ when $y = 0$ and $t_2$ when $y = L$. The shape of the curve is quite satisfactory for the technical demands of a paving slab, as figure 6 shows. The equations of the upper and lower surfaces then become

$$z = - t_1\sqrt[3]{g_1}/2 \quad \text{and} \quad z = + t_1\sqrt[3]{g_1}/2. \tag{3.5}$$

The three equations (2.10), (2.11), and (2.12) can now be rewritten as

$$\Delta(\Delta z) + \frac{1}{R^2}\left(\frac{\delta^2 z}{\delta y^2} + \mu \frac{\delta^2 z}{\delta x^2}\right) + 6c\left(\frac{\delta(\Delta z)}{\delta y} + \frac{1}{R^2}\frac{\delta z}{\delta y}\right)$$
$$+ 9c^2\left(\frac{\delta^2 z}{\delta y^2} + \mu \frac{\delta^2 z}{\delta x^2} + \frac{z}{R^2}\right) - \frac{1}{N}\left(W_1 - W_2 - \frac{P_2}{R}\right) = 0, \tag{3.6}$$

$$\Delta(\Delta z) + 6c\,\frac{\delta(\Delta z)}{\delta y} + 9c^2\left(\frac{\delta^2 z}{\delta y^2} + \mu \frac{\delta^2 z}{\delta x^2}\right) - \frac{1}{N}\left(W_1 - W_2\right) = 0, \tag{3.7}$$

$$\left(\frac{d^2}{dy^2} + 6c\,\frac{d}{cy} + 9c^2\right)\left(z'' + \frac{z}{R^2}\right) = z^{(4)} + z''/R^2 + 6c\,(z''' + z/R^2)$$
$$+ 9c^2(z'' + z/R^2) = (W_1 - W_2 - P_2/R)/N, \tag{3.8}$$

where the stiffness function $N$ is expressed by

$$N = E_1 t_1^3 g_1 \ . \tag{3.9}$$

Solutions subject to the ordinary types of boundary conditions may now be attempted for equations (3.6), 3.7), and (3.8).

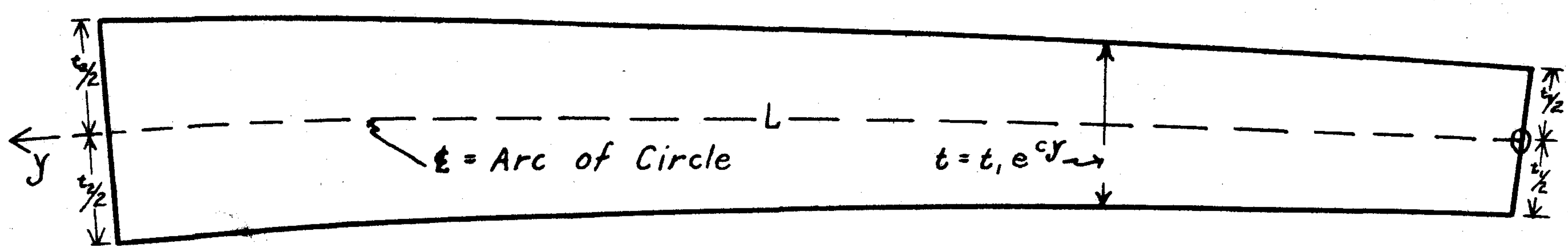



Fig. 6.

*Long. Scale 1" = 1¼'.*
*Radial Scale 1" = 2½'.*

# IV. STUDY OF THE INFINITE CURVED SLAB WITH LOAD INDEPENDENT OF THE LENGTH COORDINATE.

## 1. General Theory.

Consider a slab defined by the thickness function (3.2) of the last chapter and the limits $y = 0$, $y = L$, $x = -\infty$, $x = \infty$. If the slab is loaded in such a way that the load function $W_1$ depends upon $y$ but not upon $x$ and if $W_2 = 0$, the deflection equation takes a modified form of (2.12); e.g.,

$$\left(\frac{d^2}{dy^2} + 6c\frac{d}{dy} + 9c^2\right)\left(\frac{d^2 z}{dy^2} + \frac{z}{R^2}\right) = \frac{1}{N}\left(W_1 - \frac{P_2}{R}\right). \qquad (4.1)$$

Accompanied by the required boundary conditions, this equation would be sufficient to provide solutions for various types of loading. However, a consideration of the derivation of this equation leads to a simple method of performing two of the four integrations necessary to solve it.

If the load depends upon $y$ alone, all derivatives with respect to $x$ disappear; and thus the equilibrium equations (2.1) are

$$\frac{dP_{21}}{dy} = 0, \qquad (P_{21} = \text{Const.} = 0),$$

$$\frac{dP_2}{dy} = -\frac{Q_2}{R}, \qquad \left(Q_2 = -R\frac{dP_2}{dy}\right), \qquad (4.2)$$

$$\frac{dQ_2}{dy} = \frac{P_2}{R} - W_1, \qquad \left(P_2 = R\frac{dQ_2}{dy} + W_1 R\right).$$

The last two equations are simultaneous differential equations on $P_2$ and $Q_2$, and are satisfied if

$$P_2 = W_1R - \alpha_1 \cos y/R + \alpha_2 \sin y/R , \tag{4.3}$$

$$Q_2 = -(\alpha_1 \sin y/R + \alpha_2 \cos y/R) , \tag{4.4}$$

where $\alpha_1$ and $\alpha_2$ are arbitrary constants. Of the moment equations (2.2) only one survives; i.e.,

$$\frac{dM_2}{dy} = Q_2 \quad \text{or} \quad M_2 = \int Q_2\, dy - c_1 = R(\alpha_1 \cos y/R - \alpha_2 \sin y/R - c_1/R). \tag{4.5}$$

When this equation is combined with the definition of $M_2$ from (2.8) and the value of $Q_2$ from (4.4), the result is a second order differential equation in $z$.

$$E_1t^3(z'' + z/R^2) = -\alpha_1R \cos y/R + \alpha_2R \sin y/R + c_1. \tag{4.6}$$

The last equation can be used in place of (4.1). For the given thickness function, the solution of (4.6) has the form

$$z(y) = (a_5 + a_1 \sin y/R + a_2 \cos y/R)/g_1 + a_3 \sin y/R + a_4 \cos y/R, \tag{4.7}$$

where

$$\alpha_1 = 3cE_1t_1^3(2a_1 - 3cRa_2)/R^2, \quad \alpha_2 = 3cE_1t_1^3(2a_2 + 3cRa_1)/R^2, \quad c_1 = E_1t_1^3(9c^2R^2 + 1)/R^2. \tag{4.8}$$

In terms of these new constants, the moments and stresses may be expressed as follows:

$$z' = (1/g_1)\left[(-3ca_1 - a_2/R) \sin y/R + (a_1/R - 3ca_2) \cos y/R - 3ca_5\right] + (1/R)(a_3 \cos y/R - a_4 \sin y/R). \tag{4.9}$$

$$z'' = (1/g_1)\left[9c^2 a_1 + 6ca_2/R - a_1/R^2) \sin y/R + (9c^2 a_2 - 6ca_1/R - a_2/R^2) \cos y/R + 9c^2 a_5\right] - (1/R^2)(a_3 \sin y/R + a_4 \cos y/R). \tag{4.10}$$

$$M_2(y) = -N(z'' + z/R^2) = -E_1 t_1^3\left[(9c^2 a_1 + 6ca_2/R) \sin y/R + (9c^2 a_2 - 6ca_1/R) \cos y/R + a_5(1 + 9c^2 R^2)/R^2\right]. \tag{4.11}$$

$$Q_2(y) = -(3cE_1 t_1^3/R^2)\left[(2a_1 - 3cRa_2) \sin y/R + (2a_2 + 3cRa_1) \cos y/R\right]. \tag{4.12}$$

$$P_2(y) = W_1 R + (3cE_1 t_1^3/R^2)\left[(3cRa_2 - 2a_1) \cos y/R + (3cRa_1 + 2a_2) \sin y/R\right]. \tag{4.13}$$

## 2. Uniformly Loaded, Simply Supported Slab.

In the case of the uniformly loaded, simply supported slab, the boundary conditions are

$$z(0) = z(L) = M_2(0) = M_2(L) = 0. \tag{4.14}$$

When applied to (4.7) and (4.11), these boundary conditions lead to four equations in terms of the five parameters $a_1$, $a_2$, $a_3$, $a_4$, $a_5$. The fifth condition necessary for the determination of these constants can be found by noting the fact that

$$P_2(0) = P_2(L) = 0. \tag{4.15}$$

The two conditions of (4.15) will be satisfied if

$$P_2(0) = W_1R + (3cE_1t_1^3/R^2)(3cRa_2 - 2a_1) = 0,$$

$$P_2(L) = W_1R + (3cE_1t_1^3/R^2)[(3cRa_2 - 2a_1)\cos L/R$$

$$+ (3cRa_1 + 2a_2)\sin L/R] = 0.$$

The values of $a_1$ and $a_2$ which satisfy these two equations are

$$\begin{aligned} a_1 &= \frac{W_1R^4(2/(3cR) - \tan L/(2R))}{E_1t_1^3(9c^2R^2 + 4)}, \\ a_2 &= \frac{-W_1R^4([2/(3cR)]\tan L/(2R) + 1)}{E_1t_1^3(9c^2R^2 + 4)}. \end{aligned} \tag{4.16}$$

Since $a_1$ and $a_2$ are now fixed, so also are $\alpha_1$ and $\alpha_2$ fixed.

$$\alpha_1 = W_1R, \qquad \alpha_2 = -W_1R\tan L/(2R). \tag{4.17}$$

It would seem that there are four conditions to be imposed upon the three remaining constants $a_3$, $a_4$, and $a_5$; but the two moment equations of (4.14) are not independent and can be shown to be equivalent through the values of the $\alpha$'s given by

(4.17). Thus the problem finally degenerates to the solution of the following three simultaneous equations:

$$a_5 + a_4 = -a_2,$$

$$a_5 + a_3 g \sin L/R + a_4 g \cos L/R = -a_1 \sin L/R$$

$$- a_2 \cos L/R, \qquad (4.18)$$

$$a_5(1/R^2 + 9c^2) = 6ca_1/R - 9c^2 a_2 = 3c(2a_1/R - 3ca_2)$$

$$= R\alpha_1/(E_1 t_1^3),$$

where

$$g = e^{3cL}.$$

Equations (4.18) are satisfied by

$$a_5 = \frac{W_1 R^4}{E_1 t_1^3 (9c^2 R^2 + 1)},$$

$$a_3 = \frac{W_1 R^4}{E_1 t_1^3 (9c^2 R^2 + 4) \sin L/R} \left[\cos \frac{L}{R} \left(\frac{2}{3cR} \tan \frac{L}{2R} - \frac{3}{9c^2 R^2 + 1}\right) - \frac{1}{g}\left(\frac{2}{3cR} \tan \frac{L}{2R} + \frac{3}{9c^2 R^2 + 1}\right)\right], \qquad (4.19)$$

$$a_4 = \frac{W_1 R^4}{E_1 t_1^3 (9c^2 R^2 + 4)} \left(\frac{2}{3cR} \tan \frac{L}{2R} - \frac{3}{9c^2 R^2 + 1}\right).$$

When the equations (4.16) and (4.19) are substituted in (4.7), the explicit form of the deflection function is

$$
\begin{aligned}
z(y) = \frac{W_1 R^4}{E_1 t_1^3 (9c^2R^2 + 4)} \Bigg\{ & \frac{1}{g_1} \left[ \frac{9c^2R^2 + 4}{9c^2R^2 + 1} + \left( \frac{2}{3cR} - \tan \frac{L}{2R} \right) \sin \frac{y}{R} - \left( \frac{2}{3cR} \tan \frac{L}{2R} + 1 \right) \cos \frac{y}{R} \right] \\
& + \left[ \cos \frac{L}{R} \left( \frac{2}{3cR} \tan \frac{L}{2R} - \frac{3}{9c^2R^2 + 1} \right) - \frac{1}{g} \left( \frac{2}{3cR} \tan \frac{L}{2R} + \frac{3}{9c^2R^2 + 1} \right) \right] \sin \frac{y}{R} \\
& + \left[ \frac{2}{3cR} \tan \frac{L}{2R} - \frac{3}{9c^2R^2+1} \right] \cos \frac{y}{R} \Bigg\}.
\end{aligned}
\tag{4.20}
$$

The stresses, shears, and moments (4.13), (4.12), and (4.11) are, respectively,

$$\mu P_2 = P_1 = \mu W_1 R \left\{ 1 - \cos y/R - [\tan L/(2R)][\sin y/R] \right\}, \tag{4.21}$$

$$Q_2 = W_1 R \left\{ [\tan L/(2R)] [\cos y/R] - \sin y/R \right\}, \tag{4.22}$$

$$\mu M_2 = M_1 = \mu W_1 R^2 \left\{ [\tan L/(2R)] [\sin y/R] + \cos y/R - 1 \right\}. \tag{4.23}$$

$z(y)$ contains the thickness through $t^3$ and $c$, where $c$ depends only upon the end thicknesses. It might be well to state here that, unless indicated, $c > 0$. The case $c = 0$ which corresponds to the case of a slab with uniform thickness requires a special treatment and will be taken up later. Equations (4.21), (4.22), and (4.23) do not contain the thickness, but the unit stresses which are related to them show the effects of the thickness variation in a decided form.

The maximum value of $M_2$ occurs at $y = L/2$, just as would be the case in a flat plate of uniform thickness, and is given by the following expression:

$$M_2(L/2) = W_1R^2[\sec L/(2R) - 1].$$

If the second term be expanded and only the first two terms of the expansion be retained, the resultant value is $W_1L^2/8$, the bending moment for a uniformly loaded, straight beam which is simply supported.

$Q_2$, the shear, is zero at the middle and has the following values at the ends:

$$Q_2(0) = W_1R \tan L/(2R), \qquad Q_2(L) = -W_1R \tan L/(2R).$$

Again, if the trigonometric value be expanded and all terms which, when multiplied by $W_1R$ give resultant terms of order $\leq -1$ in R, be neglected, the result $W_1L/2$ is the same as for a beam. Of course this procedure means merely that the curvature of the slab is neglected, and should lead to such a result, as the flexure of the slab is independent of $x$.

$P_2$ is a maximum at the middle of the slab where it is given by

$$P_2(L/2) = W_1R[1 - \sec L/(2R)].$$

The unit stresses and strains become

$$\mu\sigma_2 = \sigma_1 = -\mu(P_2/t \pm 6M_2/t^2) = (-W_1R\mu/t)(1 \mp 6R/t)[1$$

$$- \cos y/R - (\tan L/2R)(\sin y/R)],$$

$$E\varepsilon_2 = (\sigma_2 - \mu\sigma_1) = \sigma_2(1 - \mu^2),$$

$$E\varepsilon_1 = (\sigma_1 - \mu\sigma_2) = 0.$$

### 3. Uniformly Loaded Slab with Clamped Edges.

In this case, the boundary conditions are

$$z(0) = z(L) = z'(0) = z'(L) = 0. \qquad (4.24)$$

The fifth condition necessary for the solution can be found by a means which could also have been used in the previous section. This condition arises from the fact that the vertical component of the load must equal the sum of the vertical components of the reactions for any one unit rib cut from the slab (figure 7).

The total vertical component of the load is

$$V = 2\int_0^{\frac{L}{2}} v ds = 2W_1 R \int_0^{\frac{L}{2R}} \cos\theta \, d\theta$$

$$= 2W_1 R \sin L/(2R).$$

Let the total vertical component of the reactions $Q_2(0)$ and $[-Q_2(L)]$ be $[v(0) + v(L)]$, where

$$v(i) = Q_2(i) \cos L/(2R),$$

$$(i = 0, L).$$

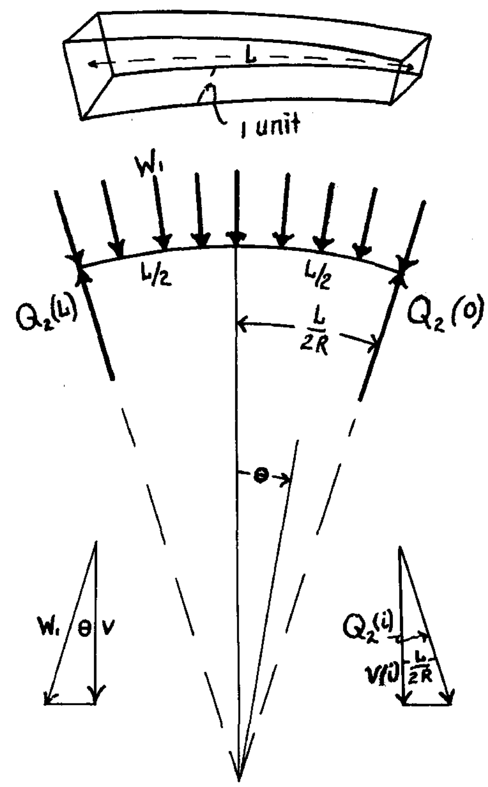



Fig. 7.

Then

$$v(0) + v(L) = [Q_2(0) - Q_2(L)] \cos L/(2R)$$

$$= [\alpha_1 \sin L/R + \alpha_2 (\cos L/R - 1)] \cos L/(2R)$$

$$= 2W_1 R \sin L/(2R),$$

and

$$\alpha_1 = \alpha_2 \tan L/(2R) + \frac{2W_1 R}{1 + \cos L/R}. \qquad (4.25)$$

When this value of $\alpha_1$ is put in equation (4.7), through the values of $a_1$ and $a_2$ in terms of $\alpha_1$ and $\alpha_2$ from (4.8), a new equation of the following form results:

$$z(y) = \frac{1}{g_1}\left\{[A \sin y/R + B \cos y/R]\alpha_2 + S \sin y/R + T \cos y/R + a_5\right\} + a_3 \sin y/R + a_4 \cos r/R, \tag{4.26}$$

where

$$A = \frac{R^3[(2/3cR) \tan p/2 + 1]}{E_1 t_1^3 (9c^2R^2 + 4)}, \quad B = \frac{R^3[(2/3cR) - \tan p/2]}{E_1 t_1^3 (9c^2R^2 + 4)},$$

$$S = \frac{4W_1R^4}{3cE_1 t_1^3 (9c^2R^2 + 4)(1 + \cos p)}, \quad T \quad \frac{2W_1R^4}{E_1 t_1^3 (9c^2R^2 + 4)(1 + \cos p)}, \tag{4.27}$$

$$p = \frac{L}{R}, \qquad g = e^{3cL}.$$

Differentiation of (4.26) leads to the slope and moment expression:

$$z' = (1/g_1)\left\{\left[(-3cA - B/R) \sin y/R + (A/R - 3cB) \cos y/R\right]\alpha_2 - (T/R + 3cS) \sin y/R + (S/R - 3cT) \cos y/R - 3ca_5\right\} + (a_3/R) \cos y/R - (a_4/R) \sin y/R. \tag{4.28}$$

$$M_2 = E_1 t_1^3 \Big\{ \Big[(9c^2 A + 6cB/R)\sin y/R + (9c^2 B - 6cA/R)\cos y/R\Big]\alpha_2 + (9c^2 S + 6cT/R)\sin y/R + (9c^2 T - 6cS/R)\cos y/R + a_5(9c^2 + 1/R^2)\Big\} . \tag{4.29}$$

When the boundary conditions (4.24) are applied to (4.26) and (4.29), the four following equations in the parameters $\alpha_2$, $a_3$, $a_4$, $a_5$ result:

$$\left.\begin{aligned} & B\alpha_2 + a_5 + a_4 = -T, \\ & (A - 3cBR)\alpha_2 - 3cRa_5 + a_3 = 3cTR - S, \\ & (A\sin p + B\cos p)\alpha_2 + a_5 + a_3 g\sin p + a_4 g\cos p \\ & \quad = -S\sin p - T\cos p, \\ & [-(B + 3cAR)\sin p + (A - 3cBR)\cos p]\alpha_2 - 3cRa_5 \\ & \quad + a_3 g\cos p - a_4 g\sin p = (T + 3cSR)\sin p \\ & \quad + (3cTR - S)\cos p. \end{aligned}\right\} \tag{4.30}$$

These equations are satisfied by the following values:

$$\alpha_1 = hW_1 \tan p/2 \Big\{ g[(4 \sin p)/(3cR) - (6 - 4 \cos p)(1 + \cos p) + 6cR \sin p\,(1 - \cos p) - 18c^2R^2 \sin^2 p] + 2 \cos p - (4 \sin p)/(3cR) + 2g^2 \Big\},$$

$$\alpha_2 = hW_1 \tan p/2 \Big\{ g[4(1 + \cos p)/(3cR) + \sin p(6 - 4 \cos p) + 6cR \sin^2 p + 18c^2R^2 \sin p(1 - \cos p)] - 2 \sin p - (4 \cos p)/(3cR) - (4g^2)/(3cR) \Big\},$$

$$a_3 = \frac{2hW_1R \tan p/2}{9c^2E_1t_1^3} \Big\{ g[3cR \cos p\,(1 - \cos p) - \sin p] + 3cR(1 - \cos p) + \sin p \Big\}, \tag{4.31}$$

$$a_4 = \frac{2hW_1R(\tan p/2)(1 - \cos p)}{9c^2E_1t_1^3} \Big\{ g(1 - 3cR \sin p) - 1 \Big\},$$

$$a_5 \equiv \frac{2hW_1R \tan p/2}{9c^2E_1t_1^3} \Big\{ - g(2 + 9c^2R^2\sin^2 p) + g^2 + 1 \Big\},$$

where

$$1/h = (g/R)[\sin p(4 \cos p - 6) + 18c^2R^2\sin p(\cos p - 1)] + (g^2/R)[2(1 - \cos p)/(3cR) + \sin p] + (1/R)[2(\cos p - 1)/(3cR) + \sin p]. \tag{4.32}$$

The solution of the clamped-edge case is now completed. The explicit function $z(y)$ is not expressed in full but is contained in (4.26), (4.27), and the constants just determined.

Then by the use of equations (4.12), (4.8), (4.31), and (4.32)

$$
\begin{aligned}
Q_2 = -W_1 h \tan p/2 \Big\{ &(\sin y/R) \Big[\Big[ 2 \cos p - (4 \sin p)/(3cR) \\
&+ g\big[(4 \sin p)/(3cR) - (6 - 4 \cos p)(1 + \cos p) \\
&+ (6cR \sin p)(1 - \cos p) - 18c^2 R^2 \sin^2 p\big] + 2g^2 \Big]\Big] \\
&+ \cos y/R \Big[\Big[ - 2 \sin p - (4 \cos p)/(3cR) + g\big[4(1 \\
&+ \cos p)/(3cR) + (\sin p)(6 - 4 \cos p) + 6cR \sin^2 p \\
&+ (18c^2 R^2 \sin p)(1 - \cos p\big] - (4g^2)/(3cR)\Big]\Big] \Big\} \quad ,
\end{aligned} \tag{4.33}
$$

$$
\begin{aligned}
Q_2(0) = -W_1 h \tan p/2 \Big\{ &-2 \sin p - (4 \cos p)/(3cR) \\
&+ g\big[4(1 + \cos p)/(3cR) + (\sin p)(6 - 4 \cos p) \\
&+ 6cR \sin^2 p + (18c^2 R^2 \sin p)(1 - \cos p)\big] - (4g^2)/(3cR) \Big\} ,
\end{aligned} \tag{4.34}
$$

$$
\begin{aligned}
Q_2(L) = -W_1 h \tan p/2 \Big\{ &- 4/(3cR) + g\big[4(1 + \cos p)/(3cR) \\
&+ (\sin p)(4 \cos p - 6) + 6cR \sin^2 p + (18c^2 R^2 \sin p) \\
&(\cos p - 1)\big] + g^2\big[2 \sin p - (4 \cos p)/(3cR)\big] \Big\} .
\end{aligned} \tag{4.35}
$$

It is interesting to note here that

$$Q_2(0) - Q_2(L) = 2W_1R \tan p/2;$$

and if $\tan p/2$ is replaced by its series expansion and $R$ is allowed to approach infinity, this algebraic sum of the reactions approaches $W_1L$, the sum of the reactions of a flat plate.

By treating the equations for moment (4.29) and stress (4.13) in a fashion similar to that in which the expression for $Q_2$ was treated, the following equations result:

$$\begin{aligned} M_2 = -W_1hR \tan p/2 \Big\{ &(\cos y/R) \Big[\Big[ -2\cos p + (4 \sin p)(3cR) \\ &+ g\big[-(4\sin p)/(3cR) + (6 - 4\cos p)(1 + \cos p) \\ &+ (6cR \sin p)(\cos p - 1) + 18c^2R^2\sin^2 p\big] - 2g^2 \Big]\Big] \\ &+ (\sin y/R) \Big[\Big[ -2\sin p - (4\cos p)/(3cR) + g\big[4(1 \\ &+ \cos p)/(3cR) + (\sin p)(6 - \cos p) + 6cR\sin^2 p \\ &+ (18c^2R^2 \sin p)(1 - \cos p)\big] - (4g^2)/(3cR) \Big]\Big] \\ &+ \big[2(9c^2R^2 + 1)/(9c^2R^2)\big]\big[g^2 - g(2 + 9c^2R^2\sin^2 p) + 1\big] \Big\}, \end{aligned} \tag{4.36}$$

$$\begin{aligned} M_2(0) = -W_1hR \tan p/2 \Big\{ &2(1 - \cos p) + (4\sin p)/(3cR) \\ &+ 2/(9c^2R^2) + g\big[2\cos p(1 - \cos p) - 4/(9c^2R^2) \\ &- (4\sin p)/(3cR) + (6cR\sin p)(\cos p - 1)\big] \\ &+ (2g^2)/(9c^2R^2) \Big\}, \end{aligned} \tag{4.37}$$

$$\begin{aligned}
M_2(L) = -W_1 hR \tan p/2 \Big\{ & 2/(9c^2R^2) + g\big[2 \cos p(1 - \cos p) \\
& + 6cR \sin p(1 - \cos p) + (4 \sin p)/(3cR) \\
& - 4/(9c^2R^2)\big] + g^2\big[2(1 - \cos p) - (4 \sin p)/(3cR) \\
& + 2/(9c^2R^2)\big]\Big\} ,
\end{aligned} \tag{4.38}$$

$$\begin{aligned}
P_2 = W_1 R - W_1 h \tan p/2 \Big\{ & (\cos y/R) \Big[\Big[2 \cos p - (4 \sin p)/(3cR) \\
& + g\big[(4 \sin p)/(3cR) + (4 \cos p - 6)(1 + \cos p) \\
& + (6cR \sin p)(1 - \cos p) - 18c^2R^2 \sin^2 p\big] + 2g^2\Big]\Big] \\
& + (\sin y/R)\Big[\Big[2 \sin p + (4 \cos p)/(3cR) + g\big[-4(1 \\
& + \cos p)/(3cR) + (\sin p)(4 \cos p - 6) - 6cR \sin^2 p \\
& + (18c^2R^2 \sin p)(\cos p - 1)\big] + (4g^2)/(3cR)\Big]\Big]\Big\} ,
\end{aligned} \tag{4.39}$$

$$\begin{aligned}
P_2(0) = W_1 h \tan p/2 \Big\{ & (2 \sin p)/(3cR) - \cos p + 1 \\
& + g\big[(6cR \sin p)(1 - \cos p) - (4 \sin p)/(3cR)\big] \\
& + g^2\big[(2 \sin p)/(3cR) + \cos p - 1\big]\Big\} ,
\end{aligned} \tag{4.40}$$

$$P_2(0) = -P_2(L). \tag{4.41}$$

If the same expansion and limiting process, which has been mentioned before, be applied to $Q_2$ and $M_2$, two formulas result which will, in the next section, be compared with the similar expressions for a beam of the same variable type of thickness as that of this plate but with plane neutral surface. These values are

$$Q_2 = W_1 \left\{ y - \frac{2}{3c} + \frac{gL(1 - 3cL + 9c^2L^2/2) - L}{g^2 - g(2 + 9c^2L^2) + 1} \right\}, \tag{4.42}$$

$$\begin{aligned} M_2 = -W_1 \Bigg\{ & \frac{y^2}{2} - \frac{2y}{3c} + \frac{1}{9c^2} + y\left[\frac{gL(1 - 3cL + 9c^2L^2/2) - L}{g^2 - g(2 + 9c^2L^2) + 1}\right] \\ & + \frac{L}{6c}\left[\frac{4 + 3cL + g(-4 + 9cL - 9c^2L^2)}{g^2 - g(2 + 9c^2L^2) + 1}\right]\Bigg\}, \end{aligned} \tag{4.43}$$

The values of $M(0)$, $M(L)$, $Q(0)$, and $Q(L)$, which could be obtained from these, are written out in the next section.

## 4. Beam with Variable Cross-Section under a Uniform Load.

Consider a beam of unit width and length $L$, whose thickness is defined by the same thickness function, $t = t_1^{3} \sqrt[3]{g_1^3}$, as was used to define the thickness of the slab. Such a beam might be thought of as a unit strip cut from the plate if the elastic surface of the plate were plane and not cylindrical.

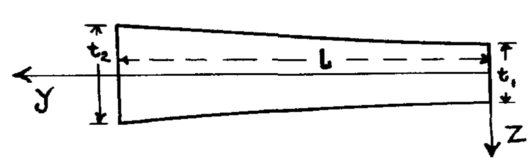



Fig. 8.

If this beam is subjected to a uniform load of $W_1$ pounds per unit of length, the equation of flexure of the beam may be written in the following manner:

$$\frac{d^2M}{dy^2} = W_1.^* \tag{4.44}$$

Two integrations with respect to $y$ will then give

$$M = W_1 y^2/2 + c_1 y + c_2. \tag{4.45}$$

Since

$$M = EIz'',^* \quad \text{where } I = t^3/12 = t_1^3 g_1/12,$$

$$z'' = [1/(E_2 g_1)][W_1 y^2/2 + c_1 y + c_2]. \tag{4.46}$$

$$E_2 = \frac{Et_1^3}{12}.$$

If equation (4.46) be integrated twice, an equation for the deflection $z$ in terms of four constants is obtained:

$$z = \frac{1}{E_2}\left\{\frac{1}{9c^2 g_1}\left[W_1\left(\frac{y^2}{2} + \frac{2y}{3c} + \frac{1}{3c^2}\right) + c_1\left(y + \frac{2}{3c}\right) + c_2\right] + c_3 y + c_4\right\}. \tag{4.47}$$

Simply Supported Beam. In the case of the simply supported beam, the boundary conditions are

$$z(0) = z(L) = M(0) = M(L). \tag{4.48}$$

*Prescott, J. Applied elasticity. p.64. Longmans, Green and Co. London. 1924.

When the last two of these are applied to (4.45), two of the constants are determined as

$$c_2 = 0 \quad \text{and} \quad c_1 = -W_1 L/2. \tag{4.49}$$

The first two conditions of (4.48) determine the other two constants as

$$\begin{aligned} c_3 &= [W_1/(27c^4 L)][1 - cL - (cL + 1)/g], \\ c_4 &= [W_1/(27c^4)][cL - 1]. \end{aligned} \tag{4.50}$$

When (4.49) and (4.50) are put in (4.47), an expression for the deflection of the simply supported beam results; i.e.,

$$\begin{aligned} z = \frac{W_1}{9c^2 E_2}\Bigg\{\frac{1}{g_1}\left[\frac{y^2}{2} + y\left(\frac{2}{3c} - \frac{L}{2}\right) + \frac{1}{3c}\left(\frac{1}{c} - L\right)\right] \\ + \frac{y}{3cL}\left[\frac{1}{c} - L - \frac{1}{g}\left(\frac{1}{c} + L\right)\right] + \frac{1}{3c}\left(L - \frac{1}{c}\right)\Bigg\}. \end{aligned} \tag{4.51}$$

A similar substitution in (4.45) will give $M$ and $Q$ as

$$M = W_1 y(y - L)/2, \tag{4.52}$$

$$Q(y) = -\frac{dM}{dy} = W_1\left(\frac{L}{2} - y\right), \quad Q(0) = \frac{W_1 L}{2}, \quad Q(L) = -\frac{W_1 L}{2}. \tag{4.53}$$

If equation (4.22) be written in the series-expansion form and the limit of this as $R \longrightarrow \infty$ be evaluated,

$$Q_2 = W_1(L/2 - y).$$

This is exactly the same as the first part of (4.53) above. In the same way (4.23) can be reduced to (4.52).

<u>Beam with Clamped Ends</u>. The case of the beam with clamped ends is more complicated than the last case, but it can be carried out in similar fashion. The boundary conditions are

$$z(0) = z(L) = z'(0) = z'(L). \tag{4.54}$$

The four equations resulting from the application of these boundary conditions follow:

$$\frac{2c_1}{27c^3} + \frac{c_2}{9c^2} + c_4 = -\frac{W_1}{27c^4},$$

$$\left(\frac{L}{9c^2} + \frac{2}{27c^3}\right)c_1 + \frac{c_2}{9c^2} + gLc_3 + gc_4 = -\frac{W_1}{9c^2}\left(\frac{L^2}{2} + \frac{2L}{3c} \quad \frac{1}{3c^2}\right),$$

$$\frac{c_1}{9c^2} + \frac{c_2}{3c} - c_3 = -\frac{W_1}{27c^3}, \tag{4.55}$$

$$\left(\frac{3cL + 1}{9c^2}\right)c_1 + \frac{c_2}{3c} - c_3 g = -\frac{W_1}{3c}\left(\frac{L^2}{2} + \frac{L}{3c} + \frac{1}{9c^2}\right).$$

These are satisfied by

$$
\begin{aligned}
c_1 &= W_1 \left\{ -\frac{2}{3c} + \frac{gL(1 - 3cL + 9c^2L^2/2) - L}{9^2 - g(2 + 9c^2L^2) + 1} \right\}, \\
c_2 &= W_1 \left\{ \frac{1}{9c^2} + \frac{L[(3cL + 4) + g(9cL - 4 - 9c^2L^2)]}{6c[g^2 - g(2 + 9c^2L^2) + 1]} \right\}, \\
c_3 &= \frac{W_1 L[g(3cL - 2) + (3cL + 2)]}{18c^2[g^2 - g(2 + 9c^2L^2) + 1]}, \\
c_4 &= \frac{W_1 L^2[g(1 - 3cL) - 1]}{18c^2[g^2 - g(2 + 9c^2L^2) + 1]}.
\end{aligned}
\tag{4.57}
$$

In terms of these constants,

$$
\begin{aligned}
M = W_1 \Bigg\{ & \frac{y^2}{2} + y\left[ -\frac{2}{3c} + \frac{gL(1 - 3cL + 9c^2L^2/2) - L}{g^2 - g(2 + 9c^2L^2) + 1} \right] \\
& + \frac{1}{9c^2} + \frac{L[3cL + 4 + g(9cL - 4 - 9c^2L^2)]}{6c[g^2 - g(2 + 9c^2L^2) + 1]} \Bigg\}.
\end{aligned}
\tag{4.58}
$$

Except for sign, (4.58) agrees with (4.43), the expression for $M_2$ in the clamped edge plate. This difference is due merely to the difference in assumption as to the direction of positive moments in the two cases. The end values of the moments are

$$
M(0) = W_1 \left\{ \frac{1}{9c^2} + \frac{L[3cL + 4 + g(9cL - 4 - 9c^2L^2)]}{6c[g^2 - g(2 + 9c^2L^2) + 1]} \right\}, \tag{4.59}
$$

$$M(L) = W_1 \left\{ \frac{1}{9c^2} + \frac{L[g^2(3cL - 4) + g(4 + 9cL + 9c^2L^2)]}{6c[g^2 - g(2 + 9c^2L^2) + 1]} \right\}. \quad (4.60)$$

The total vertical shear at any point, and thus the reaction at the ends, can be found by taking the negative of the derivative of the moment with respect to $y$:

$$Q = -\frac{dM}{dy} = -W_1 y - c_1 = -W_1 \left\{ y - \frac{2}{3c} + \frac{gL(1 - 3cL + 9c^2L^2/2) - L}{g^2 - g(2 + 9c^2L^2) + 1} \right\}, \quad (4.61)$$

$$Q(0) = +W_1 \left[ \frac{2}{3c} - \frac{gL(1 - 3cL + 9c^2L^2/2) - L}{g^2 - g(2 + 9c^2L^2) + 1} \right], \quad (4.62)$$

$$Q(L) = +W_1 \left[ \frac{2}{3c} + \frac{gL(1 + 3cL + 9c^2L^2) - Lg^2}{g^2 - g(2 + 9c^2L^2) + 1} \right]. \quad (4.63)$$

The value of $Q$ in (4.61) is the same as the value of $Q_2$ in (4.42) of the last section. Other checks upon the correctness of these results are the facts that the sum of the forces in the vertical direction and the sum of the moments acting upon the beam are each zero:

$$Q(0) - Q(L) + W_1 L = 0,$$

$$M(0) - M(L) - W_1 L^2/2 + L[-Q(L)] = 0.$$

The expression for the deflection is given by the following equation:

$$z = \frac{W_1}{18c^2 E_2}\left\{\frac{y^2}{g_1} + \frac{Lg(2 - 6cL + 9c^2L^2)y}{g_1[g^2 - g(2+9c^2L^2)+1]} + \frac{L^2[1+g(3cL-1)]}{g_1[g^2-g(2+9c^2L^2)+1]} + \frac{Ly[g(3cL-2)+3cL+2]}{g^2 - g(2+9c^2L^2)+1} + \frac{L^2[g(1-3cL)-1]}{g^2-g(2+9c^2L^2)+1}\right\}. \tag{4.64}$$

## 5. Effect of Curvature in the Case of the Simply Supported Infinite Plate.

In the case of the simply supported infinite plate with the dimensions suggested in the first section of this chapter, it is very easy to show that the curvature adds little to the shears, stresses, and moments. In the following table are listed the different elements of the solution of the plate problem for two cases. The left hand side of the table refers to a plate in which both curvature and variation of section are considered. In the right hand column only variation of section is considered.

$(L = 10, \quad R = 100, \quad t_1 = 1/2, \quad t_2 = 3/4)$

| Plate with Curvature and Variable Thickness. | Plate with Variable Thickness. |
|---|---|
| $P_2 = W_1R\left\{1 - \cos y/R - [\tan L/(2R)][\sin y/R]\right\}$ | $P_2 = 0$ |
| $P_2(\text{Max.}) = P_2(5) = -.12513W_1$ | $P_2(\text{Max.}) = 0$ |
| $M_2 = W_1R^2\left\{-1 + \cos y/R + [\tan L/(2R)][\sin y/R]\right\}$ | $M_2 = -W_1y(y - L/2)$ |
| $M_2(\text{Max.}) = M_2(5) = 12.513W_1$ | $M_2(\text{Max.}) = M_2(5) = 12.5W_1$ |
| $Q_2 = W_1R\left\{[\tan L/(2R)][\cos y/R] - \sin y/R\right\}$ | $Q_2 = W_1(L - 2y)/2$ |
| $Q_2(0) = 5.004171W_1$ | $Q_2(0) = 5W_1$ |
| $Q_2(10) = -5.004171W_1$ | $Q_2(10) = -5W_1$ |
| $\tau_{21}(0) = 10.008342W_1$ | $\tau_{21}(0) = 10W_1$ |
| $\tau_{21}(10) = 6.672228W_1$ | $\tau_{21}(10) = 6.666667W_1$ |
| $\sigma_2(\text{Surf.}) = -P_2/t \mp 6M_2/t^2$ | $\sigma_2(\text{Surf.}) = \mp 6M_2/t^2$ |
| $\sigma_2(5) = -W_1(.204 \pm 200.2)$ | $\sigma_2(5) = \mp 200W_1$ |

The next chapter deals with the simply supported plate. On the basis of this table, all consideration of curvature is omitted.

A similar table could be developed for the plate with clamped edges, showing that there, too, the curvature plays but a minor role.

# V. STUDY OF THE EFFECT OF RECTANGULAR, LINE, AND POINT LOADS ON A SIMPLY SUPPORTED RECTANGULAR SLAB WITH PLANE MIDDLE SURFACE AND VARIABLE THICKNESS.

## 1. Integral Method of Solution.

Boundary Value Problem. Consider a simply supported slab with a plane rectangular middle surface limited by the values $x = 0$, $x = a$, $y = 0$, $y = L$. Let the thickness be defined by the thickness function (3.2). Assume that a uniform load of magnitude $p_1$ acts normal to a rectangular region $B_1$ of the middle surface, where $B_1$ is defined by the dimensions $(y_2 - y_1)$ and $(x_2 - x_1)$.

Then

$$W_1(x,y) = \begin{cases} p_1 \text{ in } B_1 \\ 0 \text{ elsewhere} \end{cases} \qquad (5.1)$$

$= \text{load/unit surface.}$

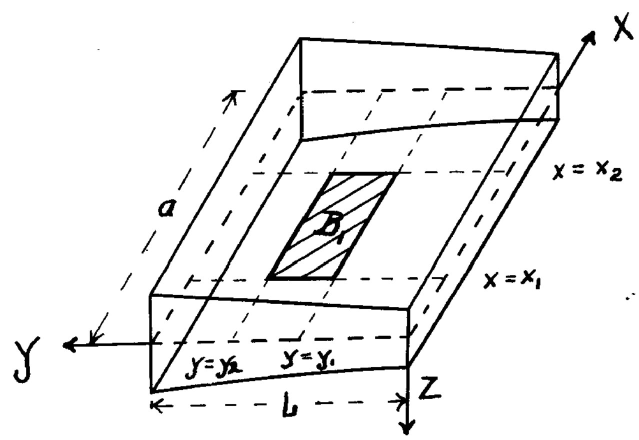



Fig. 9.

The deflection function $z(x,y)$, now slightly more complicated than (4.1), for the deflection is a function of both $x$ and $y$ instead of being a function of $y$ only, takes the following form:

$$L(z) = \Delta(\Delta z) + 6c\,\frac{\delta(\Delta z)}{\delta y} + 9c^2\left(\frac{\delta^2 z}{\delta y^2} + \mu\,\frac{\delta^2 z}{\delta x^2}\right) = \frac{W_1}{N} = \frac{W_1}{E_1 t_1^3 g_1} \quad . \qquad (5.2)$$

The boundary conditions to which the slab is subjected are

$$z\,(0,y) = z(a,y) = \Delta z(0,y) = \Delta z(a,y) = 0, \qquad (5.3)$$

$$z\,(x,0) = z(x,L) = \Delta z(x,0) = \Delta z(x,L) = 0. \qquad (5.4)$$

Thus the problem consists in finding an explicit function $z$ which will satisfy the deflection equation (5.2) and the boundary conditions (5.3) and (5.4).

Before any attempt can be made to solve this problem, some expression must be found for $W_1(x,y)$ which will satisfy conditions (5.1). In his study of the uniformly thick plate, H. Schmidt* has successfully used an integral expression for the purpose of representing a similar type of loading. Due to the fact that the kind of thickness function employed here causes $L(z)$ to have constant coefficients, the same expression can be used in this case. The basis for this expression comes from the field of complex function theory, but Schmidt has worked it out in detail in one of his articles.** The value of $W_1$ used by him depends upon the following relation:

$$f(x) = \frac{1}{2\pi i}\int_{a_0-i\infty}^{a_0+i\infty} \frac{e^{xz}}{z}\,dz = \begin{cases} 1 \text{ for } x > 0 \\ 0 \text{ for } x < 0 \end{cases}, \qquad (5.5)$$

*Schmidt, H. Theorie der Beigungschwingungen frei aufliegender Rechteckplatten unter dem Einfluss beweglicher, zeitlich periodisch veränderlicher Belastungen. Ingenieur Archiv, 2:452. 1931.

**Schmidt, H. Zur Theorie der erzwungenen Schwingungen. Zs. f. Phys., 39:474-489. 1926.

where $a_o$ is an arbitrary number greater than zero and $x$ is a parameter. By superposition of two integrals of the type (5.5) in two orthogonal directions, $W_1$ may be expressed as:

$$W_1 = p_1 \left\{ \frac{1}{2\pi i} \int_{a_o - i\infty}^{a_o + i\infty} \frac{e^{(x-x_1)n}}{n} dn \right. \tag{5.6}$$

$$\left. - \frac{1}{2\pi i} \int_{a_o - i\infty}^{a_o + i\infty} \frac{e^{(x-x_2)n}}{n} dn \right\} \left\{ \frac{1}{2\pi i} \int_{\alpha_o - i\infty}^{\alpha_o + i\infty} \frac{e^{(y-y_1)\beta}}{\beta} d\beta - \frac{1}{2\pi i} \int_{\alpha_o - i\infty}^{\alpha_o + i\infty} \frac{e^{(y-y_2)\beta}}{\beta} d\beta \right\}.$$

That this satisfies (5.1) can be seen quite clearly if one considers their product. The first brace is

$$\left\{ \begin{array}{lll} 0 - 0 = 0 & \text{for} & 0 < x < x_1 \\ 1 - 0 = 1 & \text{for} & x_1 < x < x_2 \\ 1 - 1 = 0 & \text{for} & x_2 < x < a \end{array} \right\}.$$

The second brace is

$$\left\{ \begin{array}{lll} 0 - 0 = 0 & \text{for} & 0 < y < y_1 \\ 1 - 0 = 1 & \text{for} & y_1 < y < y_2 \\ 1 - 1 = 0 & \text{for} & y_2 < y < L \end{array} \right\}.$$

Hence,

$$W_1 = p_1 \text{ (brace 1)(brace 2)} = \left\{ \begin{array}{l} p_1 \text{ in } B_1 \\ 0 \text{ elsewhere} \end{array} \right\}.$$

Solution of the Boundary Value Problem. When the right hand member of (5.2) is replaced by its integral representation (5.6), the principle of superposition of integrals of type (5.5) suggests that one may seek a solution $z(x,y)$ of this boundary value problem in the form

$$z(x,y) = \frac{P_1}{N}\,\frac{1}{2\pi i}\int_{a_o-i\infty}^{a_o+i\infty}\frac{e^{-x_1 n}-e^{-x_2 n}}{n}\Big[J_1(x,y:n)-J_2(x,y:n)\Big]dn,$$

where

$$J_s(x,y:n) = \frac{1}{2\pi i}\int_{\alpha_o-i\infty}^{\alpha_o+i\infty}\frac{e^{-y_s\beta}}{\beta}\,u(x,y:n,\beta)d\beta; \qquad (s=1,2). \tag{5.7}$$

The equation (5.2) may be replaced by a new partial differential equation:

$$D(u) = \Delta(\Delta u) - 6c\,\frac{\delta(\Delta u)}{\delta y} + 9c^2\left(\frac{\delta^2 u}{\delta y^2} + \mu\,\frac{\delta^2 u}{\delta x^2}\right) = e^{xn+y\beta}. \tag{5.8}$$

A particular solution of (5.8) is of the form

$$u_1 = \frac{e^{xn+y\beta}}{A}, \quad \text{where } A = (n^2+\beta^2)^2 - 6c\beta(n^2+\beta^2) + 9c^2(\beta^2+\mu n^2). \tag{5.9}$$

The substitution

$$u_o^{(1)} = \frac{e^{y\beta}}{A}\,f(x:n,\beta) \tag{5.10}$$

will separate the variables $x$ and $y$ in the homogeneous form of differential equation (5.8) and will produce still another differential equation:

$$\frac{d^4 f}{dx^4} + (2\beta^2 - 6c\beta + 9c^2\mu)\frac{d^2 f}{dx^2} + (\beta^4 - 6c\beta^3 + 9c^2\beta^2)f = 0. \qquad (5.11)$$

This is an ordinary fourth order differential equation and has the solution

$$f = a_1 e^{m_1 x} + a_2 e^{-m_1 x} + a_3 e^{m_2 x} + a_4 e^{-m_2 x},$$

where

$$m_1 = \sqrt{R + 3cs}\ , \qquad m_2 = \sqrt{R - 3cs}\ , \qquad (5.12)$$

$$R = \frac{-2\beta^2 + 6c\beta - 9c^2\mu}{2}, \qquad s = \frac{\sqrt{9c^2\mu + 4\beta^2\mu - 12c\beta\mu}}{2}.$$

Thus, one form of the solution of (5.8) is

$$u = u_1 + u_o^{(1)} = (e^{y\beta}/A)(e^{xn} + a_1 e^{m_1 x} + a_2 e^{-m_1 x} + a_3 e^{m_2 x} + a_4 e^{-m_2 x}). \qquad (5.13)$$

This pseudo deflection expression could also be found in terms of hyperbolic functions, but there seems to be no apparent gain in using such expressions.

In terms of u, the boundary conditions (5.3) and (5.4) become

$$u(0,y) = u(a,y) = \left[\Delta - 6c\frac{\delta}{\delta y} + 9c^2\right]u(0,y) = \left[\Delta - 6c\frac{\delta}{\delta y} + 9c^2\right]u(a,y) = 0, \qquad (5.14)$$

$$u(x,0) = u(x,L) = \left[\Delta - 6c\,\frac{\delta}{\delta y} + 9c^2\right]u(x,0)$$
$$= \left[\Delta - 6c\,\frac{\delta}{\delta y} + 9c^2\right]u(x,L) = 0. \qquad (5.15)$$

The expression in brackets can be modified slightly as the last term can be omitted because of the first two conditions in each of (5.14) and (5.15). In the case of (5.14), the value of $u$ is such that if

$$u(0,y) = u(a,y) = 0,$$

then

$$\frac{\delta}{\delta x}\left[u(0,y)\right] + \frac{\delta}{\delta y}\left[u(a,y)\right] = 0.$$

Hence, (5.14) may be rewritten in the following form:

$$u(0,y) = u(a,y) = \Delta u(0,y) = \Delta u(a,y) = 0. \qquad (5.14')$$

These boundary conditions (5.14') produce four equations in terms of the four constants of (5.13). The equations and solutions are stated below. The value of $u$ from (5.13) can then be written in terms of these constants, but it is not necessary to give its explicit form here.

$$\begin{cases} a_1 + a_2 + a_3 + a_4 = -1. \\ a_1 e^{m_1 a} + a_2 e^{-m_1 a} + a_3 e^{m_2 a} + a_4 e^{-m_2 a} = -e^{an}. \\ m_1^2(a_1 + a_2) + m_2^2(a_3 + a_4) = -n^2. \\ m_1^2(a_1 e^{m_1 a} + a_2 e^{-m_1 a}) + m_2^2(a_3 e^{m_2 a} + a_4 e^{-m_2 a}) = -n^2 e^{an}. \end{cases}$$

$$a_1(n,\beta) = -\frac{n^2 - R + 3cs}{6cs}\left(\frac{e^{an} - e^{-m_1 a}}{e^{m_1 a} - e^{-m_1 a}}\right).$$

$$a_2(n,\beta) = \frac{n^2 - R + 3cs}{6cs}\left(\frac{e^{an} - e^{m_1 a}}{e^{m_1 a} - e^{-m_1 a}}\right).$$

$$a_3(n,\beta) = \frac{n^2 - R - 3cs}{6cs}\left(\frac{e^{an} - e^{-m_2 a}}{e^{m_2 a} - e^{-m_2 a}}\right). \tag{5.16}$$

$$a_4(n,\beta) = -\frac{n^2 - R - 3cs}{6cs}\left(\frac{e^{an} - e^{m_2 a}}{e^{m_2 a} - e^{-m_2 a}}\right).$$

$$n^2 - R \pm 3cs = n^2 + \beta^2 - 3c\beta + 9c^2\mu/2 \pm (3c/2)\sqrt{9c^2\mu + 4\beta^2\mu - 12c\beta\mu}\,.$$

Since only one set of boundary conditions (5.14) has been applied, a means must be found for applying the other set (5.15). A Fourier series expansion suggests a method for doing this. Let

$$\begin{aligned} u = u_1 + u_o^{(1)} &= (e^{y\beta}/A)(e^{xn} + f) \\ &= e^{y\beta}\sum_{v=1}^{\infty} K_v(n,\beta)\sin \alpha_v x, \end{aligned} \tag{5.17}$$

where

$$\alpha_v = \frac{v\pi}{a}, \quad (v = 1,\ 2,\ 3\ldots\ldots).$$

Then

$$K_v(n,\beta) = \frac{2\alpha_v}{a}\left[\frac{1 - (-1)^v e^{an}}{n^2 + \alpha_v^2}\right]\left[\frac{1}{(R + \alpha_v^2)^2 - 9c^2\beta^2}\right]. \tag{5.18}$$

If a new function of the form

$$u_o^{(2)}(x,y:n,\beta) = \sum_{v=1}^{\infty} K_v(n,\beta)\phi_v(y:n,\beta)\ \sin\ \alpha_v x \tag{5.19}$$

be substituted in the homogeneous differential equation $D(u) = 0$, $\phi_v$ must satisfy the following condition:

$$\frac{d^4\phi_v}{dy^4} - 6c\,\frac{d^3\phi_v}{dy^3} + (9c^2 - 2\alpha_v^2)\,\frac{d^2\phi_v}{dy^2} + 6c\alpha_v^2\,\frac{d\phi_v}{dy}$$

$$+ (\alpha_v^4 - 9c^2\mu\alpha_v^2)\phi_v = 0. \tag{5.19a}$$

Such a value of $\phi_v$ is

$$\phi_v = b_1 e^{(3c/2+\delta_1)y} + b_2 e^{(3c/2-\delta_1)y}$$

$$+ b_3 e^{(3c/2+\delta_2)y} + b_4 e^{(3c/2-\delta_2)y}\ , \tag{5.20}$$

where

$$\delta_1 = \sqrt{9c^2/4 + \alpha_v^2 + 3c\alpha_v\sqrt{\mu}}, \qquad \delta_2 = \sqrt{9c^2/4 + \alpha_v^2 - 3c\alpha_v\sqrt{\mu}}\ .$$

The $\delta$'s and $b$'s are functions of $v$. This will be understood if not indicated further henceforth. The sum of the three terms $u$, $u_o^{(1)}$, $u_o^{(2)}$ will satisfy the differential equation (5.8) and fit the boundary condition (5.14). By virtue of the four constants in each $\phi_v$, this sum

$$u = u_1 + u_o^{(1)} + u_o^{(2)} = \sum_{v=1}^{\infty} K_v(e^{y\beta} + \phi_v) \sin \alpha_v x$$

$$= \sum_{v=1}^{\infty} K_v [ e^{(3c/2+\kappa)y} + b_1 e^{(3c/2+\delta_1)y} + b_2 e^{(3c/2-\delta_1)y} \tag{5.21}$$

$$+ b_3 e^{(3c/2+\delta_2)y} + b_4 e^{(3c/2-\delta_2)y} ] \sin \alpha_v x,$$

where

$$\beta = 3c/2 + \kappa,$$

may also be made to fit the boundary conditions (5.15) in a termwise and therefore a total fashion.

In applying these boundary conditions, it is more convenient to leave the last two of (5.15) in the form in which they were first stated. Omiting the $9c^2u$ term causes the loss of a certain symmetry which makes the four equations resulting when (5.15) is combined with (5.21) more difficult to solve for the four parameters $b_1$, $b_2$, $b_3$, $b_4$. These four equations and their solutions follow:

$$b_1 + b_2 + b_3 + b_4 = -1,$$

$$e^{\delta_1 L} b_1 + e^{-\delta_1 L} b_2 + e^{\delta_2 L} b_3 + e^{-\delta_2 L} b_4 = -e^{\kappa L},$$

$$(3c/2 - \delta_1)^2 b_1 + (3c/2 + \delta_1)^2 b_2 + (3c/2 - \delta_2)^2 b_3$$

$$+ (3c/2 + \delta_2^2) b_4 = -(3c/2 - \kappa)^2,$$

$$(3c/2 - \delta_1)^2 e^{\delta_1 L} b_1 + (3c/2 + \delta_1)^2 e^{-\delta_1 L} b_2 + (3c/2 - \delta_2)^2 e^{\delta_2 L} b_3$$

$$+ (3c/2 + \delta_2)^2 e^{-\delta_2 L} b_4 = -(3c/2 - \kappa)^2 e^{\kappa L},$$

$$
\begin{aligned}
b_1 = 1/V \Big\{ & 6c\delta_2[3c\delta_1 + \delta_1^2 + 3c\kappa - \kappa^2][e^{(\kappa-\delta_1)L} + 1] \\
& + [-3c\delta_1 - 3c\delta_2 - \delta_1^2 + \delta_2^2][3c\delta_2 + \delta_2^2 + 3c\kappa - \kappa^2] \\
& [e^{(\kappa-\delta_2)L} + e^{(\delta_2-\delta_1)L}] + [3c\delta_1 - 3c\delta_2 + \delta_1^2 - \delta_2^2] \\
& [-3c\delta_2 + \delta_2^2 + 3c\kappa - \kappa^2][e^{(\kappa+\delta_2)L} + {}^{-(\delta_1+\delta_2)L}]\Big\},
\end{aligned} \tag{5.22}
$$

$$
\begin{aligned}
b_2 = 1/V \Big\{ & 6c\delta_2[3c\delta_1 - \delta_1^2 - 3c\kappa + \kappa^2][e^{(\kappa+\delta_1)L} + 1] \\
& + [-3c\delta_1 - 3c\delta_2 - \delta_2^2 + \delta_1^2][3c\delta_2 - \delta_2^2 - 3c\kappa \\
& + \kappa^2][e^{(\kappa+\delta_2)L} + e^{(\delta_1-\delta_2)L}] + [-3c\delta_1 + 3c\delta_2 + \delta_1^2 \\
& - \delta_2^2][3c\delta_2 + \delta_2^2 + 3c\kappa - \kappa^2][e^{(\kappa-\delta_2)L} + e^{(\delta_1+\delta_2)L}]\Big\},
\end{aligned} \tag{5.23}
$$

$$
\begin{aligned}
b_3 = 1/V \Big\{ & 6c\delta_1[3c\delta_2 + \delta_2^2 + 3c\kappa - \kappa^2][e^{(\kappa-\delta_2)L} + 1] \\
& + [-3c\delta_1 - 3c\delta_2 - \delta_2^2 + \delta_1^2][3c\delta_1 + \delta_1^2 + 3c\kappa \\
& - \kappa^2][e^{(\kappa-\delta_1)L} + e^{(\delta_1-\delta_2)L}] + [-3c\delta_1 + 3c\delta_2 + \delta_2^2 \\
& - \delta_1^2][-3c\delta_1 + \delta_1^2 + 3c\kappa - \kappa^2][e^{(\kappa+\delta_1)L} + e^{-(\delta_1+\delta_2)L}]\Big\},
\end{aligned} \tag{5.24}
$$

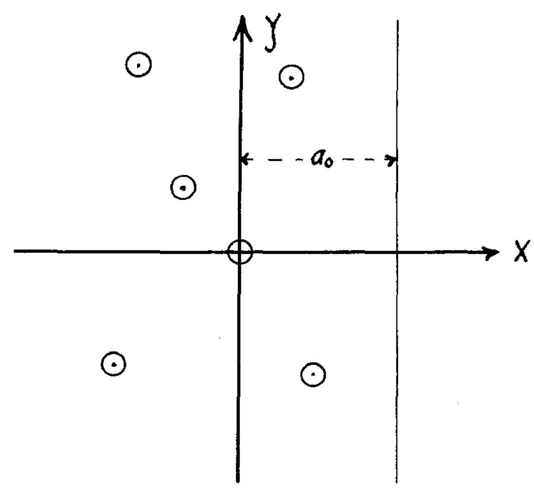
$$
\begin{aligned}
b_4 = 1/V \Big\{ & 6c\delta_1[3c\delta_2 - \delta_2^2 - 3c\kappa + \kappa^2][e^{(\kappa+\delta_2)L} + 1] \\
& + [-3c\delta_1 - 3c\delta_2 - \delta_1^2 + \delta_2^2][3c\delta_1 - \delta_1^2 - 3c\kappa \\
& + \kappa^2][e^{(\kappa+\delta_1)L} + e^{(\delta_2-\delta_1)L}] + [3c\delta_1 - 3c\delta_2 - \delta_1^2 \\
& + \delta_2^2][3c\delta_1 + \delta_1^2 + 3c\kappa - \kappa^2][e^{(\kappa-\delta_1)L} + e^{(\delta_2+\delta_1)L}]\Big\},
\end{aligned} \tag{5.25}
$$

where

$$
\begin{aligned}
V = 4 \Big\{ & [(\delta_1^2 - \delta_2^2)^2 + 9c^2(\delta_1 + \delta_2)^2][\sinh^2(\delta_1 - \delta_2)L/2] \\
& + [(\delta_1^2 - \delta_2^2)^2 - 9c^2(\delta_1 - \delta_2)^2][\sinh^2(\delta_1 + \delta_2)L/2]\Big\}.
\end{aligned} \tag{5.26}
$$

The value of $u$, which is determined by (5.21) and these constants, satisfies the transformed deflection equation and the transformed boundary conditions. It is now necessary to regress from u to z through complex integration. The process for accomplishing this is based upon a theorem developed by H. Schmidt.*

Evaluation of Contour Integrals. Let $F(z)$, a function of the complex variable $z = (x+iy)$, be an analytic function so constituted that $\lim_{z\to\infty} |z/F(z)|$ is bounded. Assume also that $F(z)$ has merely isolated zero points, n in number, whose real parts are less than a fixed number $a_o$. Hence all the zero points are to the left of the line drawn parallel to the imaginary axis in the accompanying figure. In the neighborhood of any zero point $z_\sigma$ of the order $s_\sigma \geqq 1$, a Laurent's expansion of the following form is valid:




$$1/F(z) = \sum_{\lambda=1}^{s_\sigma} A_\lambda^{(\sigma)}/(z - z_o)^\lambda + \sum_{\kappa=0}^{\infty} B_\kappa^{(\sigma)}(z - z_o)^\kappa. \tag{5.27}$$

$$(\sigma = 1, 2, \ldots. n).$$

(A zero point has the order $s_\sigma$ at $z = z_\sigma$ if at that point

*Schmidt, H. Op. cit. p. 47 (**).

the function and its $(s_\sigma - 1)^{st}$ derivatives vanish). Then for any parameter $t$

$$\frac{1}{2\pi i}\int_{a_o-i\infty}^{a_o+i\infty}\frac{e^{tz}}{F(z)}\,dz = \begin{cases}\sum_{\sigma=1}^{n}\sum_{\lambda=1}^{s_\sigma} e^{tz_\sigma} A_\lambda^{(\sigma)}\frac{t^{\lambda-1}}{(\lambda-1)!} \text{ for } t>0\\[2ex] 0 \text{ for } t<0.\end{cases} \tag{5.28}$$

Before attempting to use this theorem, it seems worth while to reiterate the salient features of the solution:

$$z = \frac{p_1}{N}\left\{\frac{1}{2\pi i}\int_{a_o-i\infty}^{a_o+i\infty}\frac{e^{-x_1 n} - e^{-x_2 n}}{n}\left[J_1 - J_2\right]dn.\right.$$

$$J_s = \frac{1}{2\pi i}\int_{\alpha_o-i\infty}^{\alpha_o+i\infty}\frac{e^{-y_s\beta}}{\beta}\,u\,d\beta = \frac{e^{-3cy_s/2}}{2\pi i}\int_{\alpha_o-3c/2-i\infty}^{\alpha_o-3c/2+i\infty}\frac{e^{-y_s\kappa}}{(\kappa+3c/2)}\,d\kappa\,. \quad (s = 1,2).$$

$$u = \sum_{v=1}^{\infty} K_v(n,\kappa)e^{3cy/2}(e^{\kappa y} + b_1 e^{\delta_1 y} + b_2 e^{-\delta_1 y} + b_3 e^{\delta_2 y}$$
$$+ b_4 e^{-\delta_2 y})\sin \alpha_v x.$$

$$K_v = \frac{2\alpha_v}{a}\left[\frac{1 - (-1)^v\,e^{an}}{(n^2+\alpha_v^2)(\kappa+\delta_1)(\kappa-\delta_1)(\kappa-\delta_2)(\kappa+\delta_2)}\right].$$

The constant $K_v$ is the only part of $J_s$ which contains $n$. This constant will be common to $J_1$ and $J_2$. Hence, $z$ may be rewritten as

$$z = \frac{2p_1}{Na} \sum_{v=1}^{\infty} \left\{ \frac{\alpha_v}{2\pi i} \int_{a_o - i\infty}^{a_o + i\infty} \left[ \frac{e^{-x_1 n} - e^{-x_2 n}}{n} \right] \left[ \frac{1 - (-1)^v e^{an}}{n^2 + \alpha_v^2} \, dn \right] \right.$$

$$\frac{1}{2\pi i} \int_{\alpha_o - 3c/2 - i\infty}^{\alpha_o - 3c/2 + i\infty} e^{3cy/2} \left[ e^{-y_1(3c/2+\kappa)} - e^{-y_2(3c/2+\kappa)} \right]$$

$$\left. \left[ \frac{e^{y} + b_1 e^{\delta_1 y} + b_2 e^{-\delta_1 y} + b_3 e^{\delta_2 y} + b_4 e^{-\delta_2 y}}{(\kappa + 3c/2)(\kappa - \delta_1)(\kappa + \delta_1)(\kappa - \delta_2)(\kappa + \delta_2)} \right] d\kappa \sin \alpha_v x \right\} . \tag{5.29}$$

$$= \frac{2p_1}{Na} \sum_{v=1}^{\infty} \left\{ \alpha_v M e^{3cy/2} [G_1 - G_2] \right\} \sin \alpha_v x,$$

where

$$M = \frac{1}{2\pi i} \int_{a_o - i\infty}^{a_o + i\infty} \frac{[1 - (-1)^v e^{an}][e^{-x_1 n} - e^{-x_2 n}]}{n(n^2 + \alpha_v^2)} \, dn, \tag{5.30}$$

and

$$G_s = \frac{e^{-3cy_s/2}}{2\pi i} \int_{\alpha_o - 3c/2 - i\infty}^{\alpha_o - 3c/2 + i\infty} \frac{e^{-\kappa y_s}(e^{\kappa y} + b_1 e^{\delta_1 y} + b_2 e^{-\delta_1 y} + b_3 e^{\delta_2 y} + b_4 e^{-\delta_2 y})}{(\kappa + 3c/2)(\kappa - \delta_1)(\kappa + \delta_1)(\kappa - \delta_2)(\kappa + \delta_2)} d\kappa . \tag{5.31}$$

With the aid of the theorem of (5.28), $M$ is evaluated next:

$$M = \frac{1}{2\pi i} \int_{a_o - i\infty}^{a_o + i\infty} \frac{e^{-x_1 n}}{F(n)} \, dn - \frac{1}{2\pi i} \left[ \int_{a_o - i\infty}^{a_o + i\infty} \frac{e^{-x_2 n}}{F(n)} \, dn - \int_{a_o - i\infty}^{a_o + i\infty} \frac{e^{(a - x_1)n}}{F(n)} \, dn (-1)^v \right]$$

$$+ \frac{(-1)^v}{2\pi i} \int_{a_o - i\infty}^{a_o + i\infty} \frac{e^{(a - x_2)n}}{F(n)} \, dn.$$

The $F(n) = n(n^2 + \alpha_v^2) = n(n + i\alpha_v)(n - i\alpha_v)$ satisfies all the conditions demanded in the theorem. $F(n)$ is analytic. $\lim_{n\to\infty} |n/F(n)| \to 0$. $F(n)$ has the isolated zero points 0, $(i\alpha_v)$, $(-i\alpha_v)$ all of which are zeros of the first order whose real parts are less than a positive $a_o$. The three Laurent's expansions in the neighborhoods of 0, $(i\alpha_v)$, and $(-i\alpha_v)$ are, respectively,

$$\frac{1}{n(n^2+\alpha_v^2)} = \frac{A_1^{(1)}}{n} + B_o^{(1)} + B_1^{(1)} n + B_2^{(1)} n^2 + \ldots\ldots ,$$

$$= \frac{A_1^{(2)}}{n-i\alpha_v} + B_o^{(2)} + B_1^{(2)}(n - i\alpha_v) + B_2^{(2)}(n - i\alpha_v)^2 + \ldots\ldots ,$$

$$= \frac{A_1^{(3)}}{(n+i\alpha_v)} + B_o^{(3)} + B_1^{(3)}(n + i\alpha_v) + B_2^{(3)}(n + i\alpha_v)^2 + \ldots\ldots ,$$

where

$$A_1^{(1)} = \left\{ n \left[\frac{1}{F(n)}\right] \right\}_{n=0} = \frac{1}{\alpha_v^2} ,$$

$$A_1^{(2)} = \left\{ (n - i\alpha_v) \left[\frac{1}{F(n)}\right] \right\}_{n=i\alpha_v} = -\frac{1}{2\alpha_v^2} ,$$

$$A_1^{(3)} = \left\{ (n + i\alpha_v) \left[\frac{1}{F(n)}\right] \right\}_{n=-i\alpha_v} = -\frac{1}{2\alpha_v^2} .$$

Because of the negative exponents, the first two integrals of M are 0. The algebraic sum of the third and fourth follows.

$$(-1)^v \left\{ -\frac{1}{2\pi i} \int_{a_o-i\infty}^{a_o+i\infty} \frac{e^{(a-x_1)n}}{F(n)}\,dn + \frac{1}{2\pi i} \int_{a_o-i\infty}^{a_o+i\infty} \frac{e^{(a-x_2)n}}{F(n)}\,dn \right\}$$

$$= (-1)^v \left\{ -\frac{1}{\alpha_v^2} + \frac{1}{2\alpha_v^2} \left( e^{i(a-x_1)\alpha_v} + e^{-i(a-x_1)\alpha_v} \right) \right.$$

$$\left. + \frac{1}{\alpha_v^2} - \frac{1}{2\alpha_v^2} \left( e^{i(a-x_2)\alpha_v} + e^{-i(a-x_2)\alpha_v} \right) \right\} .$$

Then

$$M = \frac{1}{\alpha_v^2} (\cos \alpha_v x_1 - \cos \alpha_v x_2). \qquad (5.30')$$

A critical study of $G_s$ and the values of the b's from (5.22), (5.23), (5.24), and (5.25) will show that the integrals in $G_s$ resolve themselves into four types. All the rest of the possible types of integrals disappear because of negative exponents, as happened in the evaluation of M. These four integrals are enumerated below.

$$\mu_1 = \int_{v_o-i\infty}^{v_o+i\infty} \frac{e^{(L-y_s)\kappa}\,d\kappa}{F(\kappa)} \quad . \qquad (\alpha_o - 3c/2 = v_o),$$

$$\mu_2 = \int_{v_o-i\infty}^{v_o+i\infty} \frac{e^{(L-y_s)\kappa}}{F(\kappa)}\,d\kappa \, .$$

$$\mu_3 = \int_{v_o-i\infty}^{v_o+i\infty} \varkappa^2 \frac{e^{(L-y_s)\varkappa}}{F(\varkappa)} d\varkappa.$$

$$\mu_4 = \int_{v_o-i\infty}^{v_o+i\infty} \frac{e^{(y-y_s)\varkappa}}{F(\varkappa)} d\varkappa.$$

where $F(\varkappa) = (\varkappa + 3c/2)(\varkappa - \delta_1)(\varkappa + \delta_1)(\varkappa - \delta_2)(\varkappa + \delta_2)$.

This latter function also satisfies all the conditions of the theorem and has five isolated zero points; i.e., $-3c/2$, $\delta_1$, $-\delta_1$, $\delta_2$, $-\delta_2$. In the same manner as that in which M was determined, it can be shown that

$$\begin{aligned}\mu_1 = {} & \frac{16e^{-(L-y_s)3c/2}}{(9c^2 - 4\delta_1^2)(9c^2 - 4\delta_2^2)} + \frac{2}{\delta_1(4\delta_1^2 - 9c^2)(\delta_1^2 - \delta_2^2)} \\ & \left[+ 2\delta_1 \cosh (L - y_s)\delta_1 - 3c \sinh (L - y_s)\delta_1\right] \\ & + \frac{2}{\delta_2(4\delta_2^2 - 9c^2)(\delta_1^2 - \delta_2^2)}\left[- 2\delta_2 \cosh (L - y_s)\delta_2\right. \\ & \left. + 3c \sinh (L - y_s)\delta_2\right],\end{aligned} \qquad (5.32)$$

$$\begin{aligned}\mu_2 = {} & \frac{-24ce^{-(L-y_s)3c/2}}{(9c^2 - 4\delta_1^2)(9c^2 - 4\,\delta_2^2)} + \frac{2\delta_1}{\delta_1(4\delta_1^2 - 9c^2)(\delta_1^2 - \delta_2^2)} \\ & \left[2\delta_1 \sinh (L - y_s)\delta_1 - 3c \cosh (L - y_s)\delta_1\right]\end{aligned}$$

$$+ \frac{2\delta_2}{\delta_2(4\delta_2^2 - 9c^2)(\delta_1^2 - \delta_2^2)} \Big[ - 2\delta_2 \sinh (L - y_s)\, \delta_2$$

$$+ 3c \cosh (L - y_s)\delta_2 \Big] ,$$

$$\mu_3 = \frac{36c^2 e^{-(L-y_s)3c/2}}{(9c^2 - 4\delta_1^2)(9c^2 - 4\delta_2^2)} + \frac{2\delta_1}{\delta_1(4\delta_1^2 - 9c^2)(\delta_1^2 - \delta_2^2)}$$

$$\Big[ 2\delta_1 \cosh (L - y_s)\delta_1 - 3c \sinh (L - y_s)\delta_1 \Big]$$

$$+ \frac{2\delta_2^2}{\delta_2(4\delta_2^2 - 9c^2)(\delta_1^2 - \delta_2^2)} \Big[ - 2\delta_2 \cosh (L - y_s)\delta_2$$

$$+ 3c \sinh (L - y_s)\delta_2 \Big] , \qquad (5.32) \text{ cont.}$$

$$\mu_4 = \frac{16e^{-(y-y_s)3c/2}}{(9c^2 - 4\delta_1^2)(9c^2 - 4\delta_2^2)} + \frac{2}{\delta_1(4\delta_1^2 - 9c^2)(\delta_1^2 - \delta_2^2)}$$

$$\Big[ 2\delta_1 \cosh (y - y_s)\delta_1 - 3c \sinh (y - y_s)\delta_1 \Big]$$

$$+ \frac{2}{\delta_2(4\delta_2^2 - 9c^2)(\delta_1^2 - \delta_2^2)} \Big[ - 2\delta_2 \cosh (y - y_s)\delta_2$$

$$+ 3c \sinh (y - y_s)\delta_2 \Big] .$$

$\mu_1$, $\mu_2$, and $\mu_3$ hold for any value of $y$ on the middle surface of the plate. $\mu_4$ holds only when $y > y_s$.

<u>Final Form of the Deflection Function.</u> When the $\mu$'s of (5.32) are put in (5.31), the expression for $G_s$ takes the following form:

$$G_s = \frac{16e^{-3cy/2}}{(9c^2 - 4\delta_1^2)(9c^2 - 4\delta_2^2)} + \frac{e^{-3cy_s2}}{(\delta_1^2 - \delta_2^2)} \left\{ \frac{4}{(4\delta_1^2 - 9c^2)} \right.$$

$$\left[\cosh \delta_1(y - y_s) - \frac{3c}{2\delta_1} \sinh \delta_1(y - y_s)\right] + \frac{4}{(\delta_2^2 - 9c^2)}$$

$$\left[- \cosh \delta_2(y - y_s) + \frac{3c}{2\delta_2} \sinh \delta_2(y - y_s)\right]$$

$$+ 16/V \left[9c^2\delta_1\delta_2(\cosh \delta_1(y - L) + \cosh \delta_2(y - L)\right.$$

$$- \cosh \delta_1 y \cosh \delta_2 L - \cosh \delta_2 y \cosh \delta_1 L)$$

$$+ 3c\delta_1\delta_2(\delta_1 \sinh \delta_1(y - L) + \delta_2 \sinh \delta_2(y - L)$$

$$- \delta_1 \sinh \delta_1 y \cosh \delta_2 L + \delta_2 \cosh \delta_1 y \sinh \delta_2 L$$

$$+ \delta_1 \cosh \delta_2 y \sinh \delta_1 L - \delta_2 \sinh \delta_2 y \cosh \delta_1 L)$$

$$+ (9c^2\delta_2^2 - \delta_2^4 + \delta_1^2\delta_2^2) \sinh \delta_1 y \sinh \delta_2 L + (9c^2\delta_1^2$$

$$\left.- \delta_1^4 + \delta_1^2\delta_2^2) \sinh \delta_2 y \sinh \delta_1 L\right] \left[\frac{4(\delta_1^2 - \delta_2^2)e^{-(L-y_s)3c/2}}{(9c^2 - 4\delta_1^2)(9c^2 - 4\delta_2^2)}\right.$$

$$+ \frac{1}{4\delta_1^2 - 9c^2}\left(\cosh \delta_1(L - y_s) - \frac{3c}{2\delta_1} \sinh \delta_1(L - y_s)\right)$$

$$\left.+ \frac{1}{4\delta_2^2 - 9c^2}\left(- \cosh \delta_2(L - y_s) + \frac{3c}{2\delta_2} \sinh \delta_2(L - y_s)\right)\right]$$

$$+ 16/V \left[3c\delta_2(\sinh \delta_1(y - L) - \sinh \delta_1 y \cosh \delta_2 L\right.$$

$$+ \cosh \delta_2 y \sinh \delta_1 L) + 3c\delta_1(\sinh \delta_2(y - L)$$

$$+ \cosh \delta_1 y \sinh \delta_2 L - \sinh \delta_2 y \cosh \delta_1 L) + (\delta_1^2 - \delta_2^2)$$

$$\left.(\sinh \delta_1 y \sinh \delta_2 L - \sinh \delta_2 y \sinh \delta_1 L)\right]$$

$$\Bigg[\Bigg[ - \frac{27c^2(\delta_1^2 - \delta_2^2)e^{-(L-y_s)3c/2}}{(9c^2 - 4\delta_1^2)(9c^2 - 4\delta_2^2)} + \frac{3c\delta_1}{4\delta_1^2 - 9c^2}$$

$$\left[ \sinh \delta_1(L - y_s) - \frac{3c}{2\delta_1} \cosh \delta_1(L - y_s)\right] + \frac{3c\delta_2}{4\delta_2^2 - 9c^2}$$

$$\left[ - \sinh \delta_2(L - y_s) + \frac{3c}{2\delta_2} \cosh \delta_2(L - y_s)\right] - \frac{\delta_1^2}{4\delta_1^2 - 9c^2}$$

$$\left[ \cosh \delta_1(L - y_s) - \frac{3c}{2\delta_1} \sinh \delta_1(L - y_s)\right] - \frac{\delta_2^2}{4\delta_2^2 - 9c^2}$$

$$\left[ - \cosh \delta_2(L - y_s) + \frac{3c}{2\delta_2} \sinh \delta_2(L - y_s)\right]\Bigg]\Bigg]\Bigg\} .$$

The part underlined in red holds when $y > y_s$. The rest holds for $0 \leq y \leq L$. The part underlined in blue could be omitted because it is independent of $y_s$ and will disappear in $(G_1 - G_2)$.

The deflection (5.29) of the central surface of the slab under a rectangular load now has to be expressed as

$$z = \frac{2p_1}{Na} e^{3cy/2} \sum_{v=1}^{\infty} (1/\alpha_v)(\cos \alpha_v x_1 - \cos \alpha_v x_2)[G_1 - G_2] \sin \alpha_v x = \Gamma(x,y)[G_1 - G_2] , \tag{5.33}$$

where $\Gamma(x,y)$ is the operator

$$\Gamma = \frac{2p_1}{Na} e^{3cy/2} \sum_{v=1}^{\infty} (1/\alpha_v)(\cosh \alpha_v x_1 - \cos \alpha_v x_2) \sin \alpha_v x. \tag{5.34}$$

Since the G's have parts which are not valid over the whole of the slab, a better form for z follows:

$$
\begin{aligned}
z &= \Gamma(x,y)\left[W_1^{(o)} - W_2^{(o)}\right] \quad \text{for} \quad 0 \le y \le y_1 \text{ , (Region 1)} \\
z &= \Gamma(x,y)\left[W_1^{(1)} + W_1^{(o)} - W_2^{(o)}\right] \quad \text{for} \quad y_1 \le y \le y_2 \text{ , (Region 2)} \\
z &= \Gamma(x,y)\left[W_1^{(1)} - W_2^{(2)} + W_1^{(o)} - W_2^{(o)}\right] \quad \text{for} \quad y_2 \le y \le L. \text{ (Region 3)}
\end{aligned}
\tag{5.35}
$$

(See figure 10 for definitions of regions).

$$
\begin{aligned}
&\left[W_1^{(o)} - W_2^{(o)}\right] = \\
&\left\{\frac{1}{(\delta_1^2-\delta_2^2)\left[9c^2(\delta_1+\delta_2)^2-(\delta_1^2-\delta_2^2)^2\right]\sinh^2(\delta_1-\delta_2)L/2+\left[(\delta_1^2-\delta_2^2)^2-9c^2(\delta_1-\delta_2)^2\right]\sinh^2(\delta_1+\delta_2)L/2}\right\} \\
&\quad \Big\{\Big[9c^2\delta_1\delta_2(\cosh\delta_1(y-L)+\cosh\delta_2(y-L) \\
&\quad - \cosh\delta_1 y\cosh\delta_2 L - \cosh\delta_2 y\cosh\delta_1 L \\
&\quad + 3c\delta_1^2\delta_2(\sinh\delta_1(y-L) - \sinh\delta_1 y\cosh\delta_2 L \\
&\quad + \cosh\delta_2 y\sinh\delta_1 L) + 3c\delta_2^2\delta_1(\sinh\delta_2(y-L) \\
&\quad + \cosh\delta_1 y\sinh\delta_2 L - \sinh\delta_2 y\cosh\delta_1 L) \\
&\quad + (9c^2\delta_2^2 - \delta_2^4 + \delta_1^2\delta_2^2)\sinh\delta_1 y\sinh\delta_2 L \\
&\quad + (9c^2\delta_1^2 - \delta_1^4 + \delta_1^2\delta_2^2)\sinh\delta_2 y\sinh\delta_1 L\Big] \\
&\quad \Bigg[\Bigg[\frac{4}{4\delta_1^2-9c^2}\Bigg[e^{-3cy_1/2}\left(\cosh(L-y_1)\delta_1 - \frac{3c}{2\delta_1}\sinh(L\right. \\
&\quad \left. - y_1)\delta_1\right) + e^{-3cy_2/2}\Bigg(-\cosh(L-y_2)\delta_1 \\
&\quad + \frac{3c}{2\delta_1}\sinh(L-y_2)\delta_1\Bigg)\Bigg] + \frac{4}{4\delta_2^2-9c^2}\Bigg[e^{-3cy_1/2} \\
&\quad \left(-\cosh(L-y_1)\delta_2 + \frac{3c}{2\delta_2}\sinh(L-y_1)\delta_2\right) \\
&\quad + e^{-3cy_2/2}\left(\cosh(L-y_2)\delta_2 - \frac{3c}{2\delta_2}\sinh(L-y_2)\delta_2\right)\Bigg]\Bigg] \\
&\quad + \Big[3c\delta_2\left(\sinh\delta_1(y-L) - \sinh\delta_1 y\cosh\delta_2 L\right. \\
&\quad \left. + \cosh\delta_2 y\sinh\delta_1 L\right) + 3c\delta_1\Big(\sinh\delta_2(y-L) \\
&\quad + \cosh\delta_1 y\sinh\delta_2 L - \sinh\delta_2 y\cosh\delta_1 L\Big) + (\delta_1^2 \\
&\quad - \delta_2^2)\left(\sinh\delta_1 y\sinh\delta_2 L - \sinh\delta_2 y\sinh\delta_1 L\right)\Big]
\end{aligned}
\tag{5.36}
$$

$$\left[\frac{4}{4\delta_1^2 - 9c^2}\left(\left(e^{-3cy_1/2}\left(\frac{9c\delta_1}{2}\sinh(L - y_1)\delta_1\right.\right.\right.\right.$$

$$\left.- \frac{9c^2}{2}\cosh(L - y_1)\delta_1 - \delta_1^2\cosh(L - y_1)\delta_1\right)$$

$$+ e^{-3cy_2/2}\left(-\frac{9c\delta_1}{2}\sinh(L - y_2)\delta_1 + \frac{9c^2}{2}\cosh(L-y_2)\delta_2\right.$$

$$\left.\left.\left.+ \delta_1^2\cosh(L - y_2)\delta_1\right)\right) + \frac{4}{4\delta_2^2 - 9c^2}\left(\left(e^{-3cy_1/2}\right.\right. \tag{5.36 cont.}$$

$$\left(-\frac{9c\delta_2}{2}\sinh(L - y_1)\delta_2 + \frac{9c^2}{2}\cosh(L - y_1)\delta_2\right.$$

$$\left.+ \delta_2^2\cosh(L - y_1)\delta_2\right) + e^{-3cy_2/2}\left(\frac{9c\delta_2}{2}\sinh(L-y_2)\delta_2\right.$$

$$\left.\left.\left.\left.- \frac{9c^2}{2}\cosh(L - y_2)\delta_2 - \delta_2^2\cosh(L - y_2)\delta_2\right)\right)\right]\right\}.$$

$$W_1^{(1)} = \frac{16e^{-3cy/2}}{(9c^2-4\delta_1^2)(9c^2-4\delta_2^2)} + \frac{4e^{-3cy_1/2}}{\delta_1^2 - \delta_2^2}\left\{\frac{1}{4\delta_1^2 - 9c^2}\left[\cosh(y\right.\right.$$

$$\left.- y_1)\delta_1 - \frac{3c}{2\delta_1}\sinh(y - y_1)\delta_1\right] + \frac{1}{4\delta_2^2 - 9c^2} \tag{5.37}$$

$$\left.\left[-\cosh(y - y_1)\delta_2 + \frac{3c}{2\delta_2}\sinh(y - y_1)\delta_2\right]\right\}.$$

$$W_1^{(1)} - W_2^{(2)} = \frac{4}{(4\delta_1^2 - 9c^2)(\delta_1^2 - \delta_2^2)}\left\{e^{-3cy_1/2}\left[\cosh(y-y_1)\delta_1\right.\right.$$

$$\left.- \frac{3c}{2\delta_1}\sinh(y - y_1)\delta_1\right] + e^{-3cy_2/2}\left[-\cosh(y-y_2)\delta_1\right. \tag{5.38}$$

$$+ \frac{3c}{2\delta_1} \sinh (y - y_2)\delta_1 \Big]\Big\} + \frac{4}{(4\delta_2^2 - 9c^2)(\delta_1^2 - \delta_2^2)}$$

$$\Big\{ e^{-3cy_1/2} \Big[ - \cosh (y - y_1)\delta_2 + \frac{3c}{2\delta_2} \sinh (y - y_1)\delta_2 \Big] \qquad \begin{matrix}(5.38)\\ \text{cont.}\end{matrix}$$

$$+ e^{-3cy_2/2} \Big[ \cosh (y - y_2)\delta_2 - \frac{3c}{2\delta_2} \sinh (y - y_2)\delta_2 \Big] \Big\}.$$

This completes the solution for the general case of a uniform load of rectangular area bearing upon a simply supported rectangular slab of variable thickness. On account of its generality, this solution is very cumbersome. Special cases to be taken up in later sections will show considerable simplification of these results. Before proceeding to these cases, it seems desirable to exhibit another method of obtaining the same end results for this boundary value problem.

## 2. Fourier Series Method of Solution.

Boundary Value Problem. The general form of the solution (5.35) of the deflection equation suggests another way of solving this equation. This method, as mentioned before, is similar to one used by Nadai* in his treatment of certain types of plates with uniform thickness. The plate is divided into three regions as is shown by the shading in the figure.

*Nadai, A. Elastische Platten. p. 81. J. Springer, Berlin.

Since $W_1$ is zero in regions 1 and 3, the deflection equation becomes $L(z) = 0$ in those sections. In region 2, consider a strip one unit in width extending from $x = 0$ to $x = a$. Along this strip from left to right

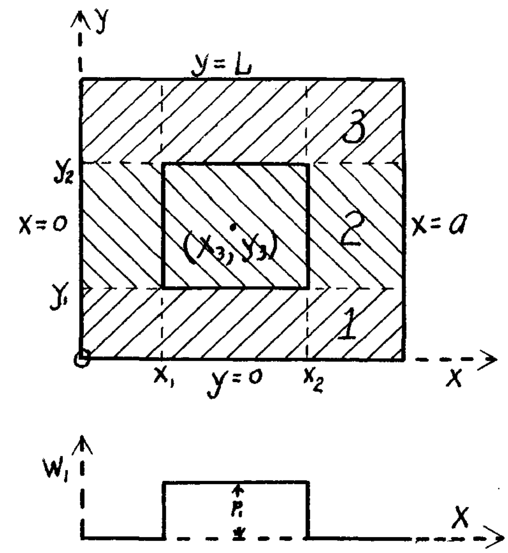



Fig. 10.

$$W_1(x) = 0 \quad \text{for} \quad 0 < x < x_1 ,$$
$$W_1(x) = p_1 \quad \text{for} \quad x_1 < x < x_2 ,$$
$$W_1(x) = 0 \quad \text{for} \quad x_2 < x < a .$$

This is a function with discontinuity of $p_1$ at $x_1$ and $x_2$. Such a function can be obtained by expanding $W_1(x)$ in a Fourier series of sines. Let

$$W_1 = \sum_{v=1}^{\infty} a_v \sin \frac{v\pi x}{a} = \sum_{v=1}^{\infty} a_v \sin \alpha_v x.$$

Then

$$a_v = (2p_1/v\pi)(\cos \frac{v\pi x_1}{a} - \cos \frac{v\pi x_2}{a}) = (2p_1/\alpha_v a)(\cos \alpha_v x_1 - \cos \alpha_v x_2),$$

and

$$W_1 = (2p_1/a) \sum_{v=1}^{\infty} (1/\alpha_v)(\cos \alpha_v x_1 - \cos \alpha_v x_2)\sin \alpha_v x \quad (5.39)$$

in region 2. This function satisfies all the necessary conditions for $W_1$ and suggests the type of solution for the differential equation.

$$L(z) = (W_1/N) = (2p_1/Na) \sum_{v=1}^{\infty} (1/\alpha_v)(\cos \alpha_v x_1 - \cos \alpha_v x_2) \sin \alpha_v x \tag{5.40}$$

as

$$z = \sum_{v=1}^{\infty} A_v \sin \alpha_v x e^{-py} f(y), \tag{5.41}$$

where $p = 3c$. Such a value of $z$ satisfies the boundary conditions (5.4) at $x = 0$ and $x = a$. When this value of $z$ is inserted in $L(z)$,

$$L(z) = \sum_{v=1}^{\infty} A_v e^{-py} \sin \alpha_v x \left\{ \frac{d^4 f}{dy^4} - 2p \frac{d^3 f}{dy^3} + (p^2 - 2\alpha_v^2) \cdot \frac{d^2 f}{dy^2} + 2p\alpha_v^2 \frac{df}{dy} + (\alpha_v^4 - \mu\alpha_v^2 p^2) f \right\} \tag{5.42}$$

$$= \frac{2p_1}{Na} \sum_{v=1}^{\infty} \frac{1}{\alpha_v} (\cos \alpha_v x_1 - \cos \alpha_v x_2) \sin \alpha_v x.$$

$L(z)$ will be satisfied if

$$A_v = (2p_1/E_1 t_1^3 a\alpha_v)(\cos \alpha_v x_1 - \cos \alpha_v x_2), \tag{5.43}$$

and if

$$D(f_v) = \frac{d^4 f}{dy^4} - 2p \frac{d^3 f}{dy^3} + (p^2 - 2\alpha_v^2) \frac{d^2 f}{dy^2} + 2p\alpha_v^2 \frac{df}{dy} + (\alpha_v^4 - \mu\alpha_v^2 p^2) f = 1. \tag{5.44}$$

The same values of $z$ and $A_v$ will be admissible in regions 1 and 3 if $D(f_v) = 0$ there.

A particular solution $D(f_v) = 1$ is

$$f = 1/(\alpha_v^4 - \mu\alpha_v^2 p^2) = U \tag{5.45}$$

The solution of $D(f_v) = 0$ is

$$f = e^{py/2}[d_1 e^{\delta_1 y} + d_2 e^{-\delta_1 y} + d_3 e^{\delta_2 y} + d_4 e^{-\delta_2 y}], \tag{5.46}$$

where the d's are constants and where $\delta_1$ and $\delta_2$ have the same values, (5.20), as in the last section. Again, single subscript notation is used although the $\delta$'s are not independent of $v$. Another relationship between this section and the last one is embodied in the fact that the differential equation (5.44) on $f_v$ is of the same form as the differential equation (5.19a) on $\phi_v$.

In the three regions 1, 2, and 3, $f_v$ takes the following respective values:

$$\begin{aligned} f_{v_1} &= e^{py/2}(a_1 e^{\delta_1 y} + a_2 e^{-\delta_1 y} + a_3 e^{\delta_2 y} + a_4 e^{-\delta_2 y}) \\ &= e^{py/2}\bar{f}_1, \\ f_{v_2} &= e^{py/2}(b_1 e^{\delta_1 y} + b_2 e^{-\delta_1 y} + b_3 e^{\delta_2 y} + b_4 e^{-\delta_2 y} \\ &\quad + Ue^{-py/2}) = e^{py/2}\bar{f}_2, \\ f_{v_3} &= e^{py/2}(c_1 e^{\delta_1 y} + c_2 e^{-\delta_1 y} + c_3 e^{\delta_2 y} + c_4 e^{-\delta_2 y}) \\ &= e^{py/2}\bar{f}_3 . \end{aligned} \tag{5.47}$$

Equations (5.41) and (5.47) give the solution for $z(x,y)$ in all three regions, if the constants $a_i$, $b_i$, $c_i$ (i=i,2,3,4) are determined from the known regional boundary relations which are outlined in the next section.

Solution of the Boundary Value Problem. On the boundary between regions which are adjoining, the deflection, slope, moment, and shear expressions which arise in the two sections must be equivalent. The relation of these expressions to the $\bar{f}$'s are given below:

$$z \sim \bar{f}$$

$$\frac{\delta z}{\delta y} \sim \frac{d\bar{f}}{dy}$$

$$My \sim \frac{d^2\bar{f}}{dy^2} - p\,\frac{d\bar{f}}{dy} + \left(\frac{p^2}{4} - \mu\alpha_v^2\right)\bar{f}\ .$$

$$Qy \sim \frac{d^3\bar{f}}{dy^3} - \frac{3p^2}{2}\,\frac{d^2\bar{f}}{dy^2} + \left(\frac{3p^2}{4} - \alpha_v^2\right)\frac{d\bar{f}}{dy} + \left(\frac{\alpha_v^2 p}{2} - \frac{p^3}{8}\right)\bar{f}.$$

Since $\bar{f}$ and $\frac{d\bar{f}}{dy}$ are to be equated by the deflection and slope conditions on the boundary between two regions, the conditions due to My and Qy may be replaced by equations of $\frac{d^2\bar{f}}{dy^2}$ and $\frac{d^3\bar{f}}{dy^3}$ on these boundaries. Hence, four equations, connecting the a's and b's of (5.47), arise on the boundary between regions 1 and 2.

$$
\begin{aligned}
&(a_1 - b_1)e^{\delta_1 y_1} + (a_2 - b_2)e^{-\delta_1 y_1} + (a_3 - b_3)e^{\delta_2 y_1} \\
&\qquad + (a_4 - b_4)e^{-\delta_2 y_1} - Ue^{-py_1/2} = 0, \\
&\delta_1(a_1 - b_1)e^{\delta_1 y_1} + \delta_1(b_2 - a_2)e^{-\delta_1 y_1} + \delta_2(a_3 - b_3)e^{\delta_2 y_1} \\
&\qquad + \delta_2(b_4 - a_4)e^{-\delta_2 y_1} + (pU/2)e^{-py_1/2} = 0, \\
&\delta_1^2(a_1 - b_1)e^{\delta_1 y_1} + \delta_1^2(a_2 - b_2)e^{-\delta_1 y_1} + \delta_2^2(a_3 - b_3)e^{\delta_2 y_1} \\
&\qquad + \delta_2^2(a_4 - b_4)e^{-\delta_2 y_1} - (p^2 U/4)e^{-py_1/2} = 0, \\
&\delta_1^3(a_1 - b_1)e^{\delta_1 y_1} + \delta_1^3(b_2 - a_2)e^{-\delta_1 y_1} + \delta_2^3(a_3 - b_3)e^{\delta_2 y_1} \\
&\qquad + \delta_2^3(b_4 - a_4)e^{-\delta_2 y_1} + (p^3 U/8)e^{-py_1/2} = 0.
\end{aligned}
\tag{5.48}
$$

Their solution follows:

$$
\begin{aligned}
a_1 - b_1 &= \frac{U(\delta_1 - p/2)(p^2/4 - \delta_2^2)e^{-y_1(p/2 + \delta_1)}}{2\delta_1(\delta_1^2 - \delta_2^2)}, \\
a_2 - b_2 &= \frac{U(\delta_1 + p/2)(p^2/4 - \delta_2^2)e^{-y_1(p/2 - \delta_1)}}{2\delta_1(\delta_1^2 - \delta_2^2)}, \\
a_3 - b_3 &= \frac{U(\delta_2 - p/2)(\delta_1^2 - p^2/4)e^{-y_1(p/2 + \delta_2)}}{2\delta_2(\delta_1^2 - \delta_2^2)}, \\
a_4 - b_4 &= \frac{U(\delta_2 + p/2)(\delta_1^2 - p^2/4)e^{-y_1(p/2 - \delta_2)}}{2\delta_2(\delta_1^2 - \delta_2^2)}.
\end{aligned}
\tag{5.49}
$$

A similar set of conditions and constants arises from the boundary between regions 2 and 3 of the plate.

$$
\begin{aligned}
c_1 - b_1 &= \frac{U(\delta_1 - p/2)(p^2/4 - \delta_2^2)e^{-y_2(p/2 + \delta_1)}}{2\delta_1(\delta_1^2 - \delta_2^2)}, \\
c_2 - b_2 &= \frac{U(\delta_1 + p/2)(p^2/4 - \delta_2^2)e^{-y_2(p/2 - \delta_1)}}{2\delta_1(\delta_1^2 - \delta_2^2)}, \\
c_3 - b_3 &= \frac{U(\delta_2 - p/2)(\delta_1^2 - p^2/4)e^{-y_2(p/2 + \delta_2)}}{2\delta_2(\delta_1^2 - \delta_2^2)}, \\
c_4 - b_4 &= \frac{U(\delta_2 + p/2)(\delta_1^2 - p^2/4)e^{-y_2(p/2 - \delta_2)}}{2\delta_2(\delta_1 - \delta_2^2)}.
\end{aligned}
\tag{5.50}
$$

At each of the edges $y = 0$ and $y = L$, two more conditions must be satisfied: e.g., $z = \Delta^2 z = 0$. This is equivalent to stating that moment and deflection are zero at those two edges. In terms of $\bar{f}$ these are

$$
\begin{aligned}
\bar{f}_1(0) = \bar{f}_3(L) &= \bar{f}_1''(0) - p\bar{f}_1'(0) + (p^2/4 - \mu\alpha^2)\bar{f}_1(0) \\
&= \bar{f}_3''(L) - p\bar{f}_3'(L) + (p^2/4 - \mu\alpha_v^2)\bar{f}_3(L) = 0.
\end{aligned}
\tag{5.51}
$$

From (5.51) four more equations arise, giving a total of twelve equations in the twelve constants or parameters $a_i$, $b_i$, $c_i$ $(i=1,2,3,4)$. These four equations are enumerated below.

$$
\begin{aligned}
& a_1 + a_2 + a_3 + a_4 = 0. \\
& (\delta_1 - p/2)^2 a_1 + (\delta_1 + p/2)^2 a_2 + (\delta_2 - p/2)^2 a_3 \\
& \qquad + (\delta_2 + p/2)^2 a_4 = 0. \\
& c_1 e^{\delta_1 L} + c_2 e^{-\delta_1 L} + c_3 e^{\delta_2 L} + c_4 e^{-\delta_2 L} = 0. \\
& (\delta_1 - p/2)^2 e^{\delta_1 L} c_1 + (\delta_1 + p/2) e^{-\delta_1 L} c_2 + (\delta_2 - p/2) e^{\delta_2 L} \\
& \qquad + (\delta_2 + p/2) e^{-\delta_2 L} c_4 = 0.
\end{aligned}
\tag{5.52}
$$

When the twelve equations (5.49), (5.50), and (5.52) are solved, the values for the constants are found to be as follows:

$$
\begin{aligned}
a_1 &= (1/V)\Big\{(\lambda^2 - \beta^2)(\tau\phi^2 - \psi)e^{-\delta_2 L} + (\beta^2 - \phi^2)(\tau\lambda^2 \\
&\quad - \psi)e^{\delta_2 L} + (\phi^2 - \lambda^2)(\tau\beta^2 - \psi)e^{-\delta_1 L}\Big\}, \\
a_2 &= (1/V)\Big\{(\lambda^2 - \phi^2)(\tau\rho^2 - \psi)e^{\delta_1 L} + (\phi^2 - \rho^2)(\tau\lambda^2 \\
&\quad - \psi)e^{\delta_2 L} + (\rho^2 - \lambda^2)(\tau\phi^2 - \psi)e^{-\delta_2 L}\Big\}, \\
a_3 &= (1/V)\Big\{(\rho^2 - \phi^2)(\tau\beta^2 - \psi)e^{-\delta_1 L} + (\phi^2 - \beta^2)(\tau\rho^2 \\
&\quad - \psi)e^{\delta_1 L} + (\beta^2 - \rho^2)(\tau\phi^2 - \psi)e^{-\delta_2 L}\Big\}, \\
a_4 &= (1/V)\Big\{(\lambda^2 - \rho^2)(\tau\beta^2 - \psi)e^{-\delta_1 L} + (\beta^2 - \lambda^2)(\tau\rho^2 \\
&\quad - \psi)e^{\delta_1 L} + (\rho^2 - \beta^2)(\tau\lambda^2 - \psi)e^{\delta_2 L}\Big\},
\end{aligned}
\tag{5.53}
$$

$$
\begin{aligned}
c_1 &= a_1 + (\theta\rho\lambda\phi/\delta_1)[e^{-y_1\beta} - e^{-y_2\beta}], \\
c_2 &= a_2 + (\theta\beta\lambda\phi/\delta_1)[e^{y_1\rho} - e^{+y_2\rho}], \\
c_3 &= a_3 - (\theta\lambda\rho\beta/\delta_2)[e^{-y_1\phi} - e^{-y_2\phi}], \\
c_4 &= a_4 - (\theta\phi\rho\beta/\delta_2)[e^{y_1\lambda} - e^{+y_2\lambda}],
\end{aligned}
\tag{5.54}
$$

$$
\begin{aligned}
b_1 &= a_1 + (\theta\rho\lambda\phi/\delta_1)e^{-y_1\beta}, \\
b_2 &= a_3 + (\theta\lambda\phi\beta/\delta_2)e^{+y_1\beta}, \\
b_3 &= a_3 - (\theta\lambda\rho\beta/\delta_2)e^{-y_1\phi}, \\
b_4 &= a_4 - (\theta\phi\rho\beta/\delta_2)e^{y_1\lambda},
\end{aligned}
\tag{5.55}
$$

where new values as explained below, have been introduced.

$$\rho = \delta_1 - p/2, \quad \beta = \delta_1 + p/2, \quad \lambda = \delta_2 - p/2,$$
$$\phi = \delta_2 + p/2, \quad \theta = U/[2(\delta_1^2 - \delta_2^2)] \, ,$$

$$\begin{aligned} \tau = \theta \Big\{ & - (\rho\lambda\phi/\delta_1)[e^{-y_1\beta} - e^{-y_2\beta}]e^{\delta_1 L} - (\beta\lambda\phi/\delta_1)[e^{y_1\rho} \\ & - e^{+y_2\rho}]e^{-\delta_1 L} + (\lambda\rho\beta/\delta_2)[e^{-y_1\phi} - e^{-y_2\phi}]e^{\delta_2 L} \\ & + (\phi\rho\beta/\delta_2)[e^{y_1\lambda} - e^{+y_2\lambda}]e^{-\delta_2 L} \Big\} , \end{aligned} \tag{5.56}$$

$$\begin{aligned} V = 4 \Big\{ & [9c^2(\delta_1 + \delta_2)^2 - (\delta_1^2 - \delta_2^2)^2] \sinh^2(\delta_1 - \delta_2)L/2 \\ & + [(\delta_1^2 - \delta_2^2)^2 - 9c^2(\delta_1 - \delta_2)^2] \sinh^2(\delta_1 + \delta_2)L/2 \Big\} , \end{aligned}$$

$$\begin{aligned} \psi = \theta \Big\{ & - (\rho^3\lambda\phi/\delta_1)[e^{-y_1\beta} - e^{-y_2\beta}]e^{\delta_1 L} - (\beta^3\lambda\phi/\delta_1)[e^{y_1\rho} \\ & - e^{y_2\rho}]e^{-\delta_1 L} + (\lambda^3\rho\beta/\delta_2)[e^{-y_1\phi} - e^{-y_2\phi}]e^{\delta_2 L} \\ & + (\phi^3\rho\beta/\delta_2)[e^{y_1\lambda} - e^{y_2\lambda}]e^{-\delta_2 L} \Big\} . \end{aligned}$$

The expression (5.41) for the deflection, when subjected to (5.53), (5.54), (5.55), and (5.56) through (5.47), leads to three regional definitions of $z$ which agree with those of the last section; i.e.,

$$z = \Gamma(x,y)[W_1^{(0)} - W_2^{(0)}] \quad \text{for} \quad 0 \le y \le y_1 \, ,$$

$$z = \Gamma(x,y)[W_1^{(1)} + W_1^{(0)} - W_2^{(0)}] \quad \text{for} \quad y_1 \le y \le y_2 \, ,$$

$$z = \Gamma(x,y)[W_1^{(1)} - W_2^{(2)} + W_1^{(0)} - W_2^{(0)}] \quad \text{for} \quad y_2 \le y \le L \, .$$

Hence, the two solutions agree. Some comparisons of the two methods will be given in the final chapter on conclusions.

## 3. Special Cases.

Uniform Load over the Whole Slab. The value of $z$ for this case can be found from the middle equation of (5.35) by letting $y_1 \to 0$, $y_2 \to L$, $x_1 \to 0$, and $x_2 \to a$. The deflection expression takes the following form:

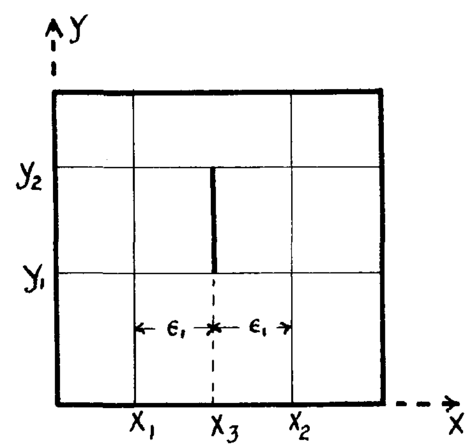
$$z = (2p_1/Na)e^{3cy/2}\sum_{v=1}^{\infty}(1/\alpha_v)[1-(-1)^v][\overline{W}_1^{(1)} + W_1^{(0)} - W_2^{(0)}]\sin\alpha_v x, \tag{5.57}$$

where

$$\begin{aligned}
[\overline{W}_1^{(1)} + W_1^{(0)} - W_2^{(0)}] = {} & \{16/[(\delta_1^2 - \delta_2^2)V]\}\{9c^2\delta_1\delta_2[\cosh\delta_1(y-L) \\
& + \cosh\delta_2(y-L) - \cosh\delta_1 y\cosh\delta_2 L - \cosh\delta_2 y\cosh\delta_1 L] \\
& + 3c\delta_1^2\delta_2[\sinh\delta_1(y-L) - \sinh\delta_1 y\cosh\delta_2 L \\
& + \cosh\delta_2 y\sinh\delta_1 L] + 3c\delta_2^2\delta_1[\sinh\delta_2(y-L) \\
& + \cosh\delta_1 y\sinh\delta_2 L - \sinh\delta_2 y\cosh\delta_1 L] + [9c^2\delta_2^2 \\
& - \delta_2^4 + \delta_1^2\delta_2^2]\sinh\delta_1 y\sinh\delta_2 L + [9c^2\delta_1^2 - \delta_1^4 \\
& + \delta_2^2\delta_1^2]\sinh\delta_2 y\sinh\delta_1 L\}\{[1/(4\delta_1^2 - 9c^2)][\cosh L\delta_1 \\
& - e^{-3cL/2} - (3c/2\delta_1)\sinh L\delta_1] + [1/(4\delta_2^2 - 9c^2)] \\
& [-\cosh L\delta_2 + e^{-3cL/2} + (3c/2\delta_2)\sinh L\delta_2] \\
& + \{16/[(\delta_1^2 - \delta_2^2)V]\}\{3c\delta_2[\sinh\delta_1(y-L) \\
& - \sinh\delta_1 y\cosh\delta_2 L + \cosh\delta_2 y\sinh\delta_1 L] \\
& + 3c\delta_1[\sinh\delta_2(y-L) + \cosh\delta_1 y\sinh\delta_2 L \\
& - \sinh\delta_2 y\cosh\delta_2 L] + [\delta_1^2 - \delta_2^2][\sinh\delta_1 y\sinh\delta_2 L \\
& - \sinh\delta_2 y\sinh\delta_1 L]\}\{[1/(4\delta_1^2 - 9c^2)][(9c\delta_1/2)\sinh L\delta_1 \\
& - (9c^2/2)(\cosh L\delta_1 - e^{-3cL/2}) - \delta_1^2(\cosh L\delta_1 \\
& - e^{-3cL/2})] + [1/(4\delta_2^2 - 9c^2)][-(9c\delta_2/2)\sinh L\delta_2
\end{aligned} \tag{5.58}$$

$$+ (9c^2/2)(\cosh L\delta_2 - e^{-3cL/2}) + \delta_2^2(\cosh L\delta_2$$
$$- e^{-3cL/2})]\} + [16/(9c^2 - 4\delta_1^2)(9c^2 - 4\delta_2^2)]e^{-3cy/2}$$
$$+ [4/(4\delta_1^2 - 9c^2)(\delta_1^2 - \delta_2^2)][\cosh y\delta_1 - (3c/2\delta_1)\sinh y\delta_1] \quad \text{(5.58) cont.}$$
$$+ [4/(4\delta_2^2 - 9c^2)(\delta_1^2 - \delta_2^2)][- \cosh y\delta_2$$
$$+ (3c/2\delta_2)\sinh y\delta_2] .$$

Line Loads. The deflection due to a line load parallel to either edge of the slab may be obtained from the deflection due to a rectangular load by letting one of the dimensions of the rectangular load approach zero. In case the line load is parallel to the Y-axis (see figure), the deflection can be found very readily by letting $x_1 = x_3 - \varepsilon_1$, $x_2 = x_3 + \varepsilon_1$, $p_1 = p_2/2\varepsilon_1$ in equation (5.35), and then taking the limit of the resultant expression as $\varepsilon_1 \rightarrow 0$. $p_2$ is the load per unit length of line. This process leads to the following three equations:




$$z = \frac{2p_2}{Na}\sum_{v=1}^{\infty}\left(\sin \alpha_v x_3 \sin \alpha_v x\right)e^{3cy/2}[\overline{W}_1^{(o)} - W_2^{(o)}]$$

$$\text{for} \quad 0 \leq y \leq y_1 \quad ,$$

$$\text{(5.59)}$$

$$z = \frac{2p_2}{Na}\sum_{v=1}^{\infty}\left(\sin \alpha_v x_3 \sin \alpha_v x\right)e^{3cy/2}[\overline{W}_1^{(1)} + W_1^{(o)} - W_2^{(o)}]$$

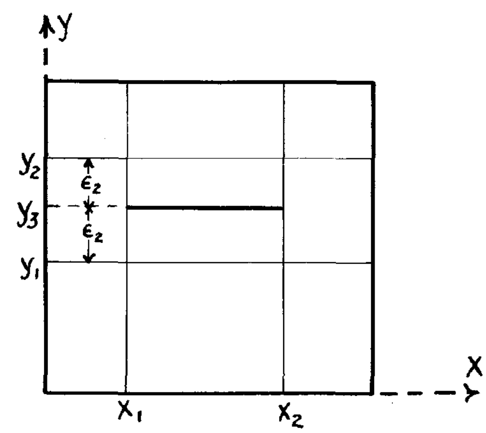
$$\text{for} \quad y_1 \leq y \leq y_2 \quad ,$$

$$z = \frac{2p_2}{Na} \sum_{v=1}^{\infty} (\sin \alpha_v x_3 \, \sin \alpha_v x) e^{3cy/2} [\overline{W}_1^{(1)} - W_2^{(2)} + W_1^{(0)} - W_2^{(0)}] \qquad (5.59) \text{ cont.}$$

for $y_2 \leq y \leq L$ .

The values for $[\overline{W}_1^{(0)} - W_2^{(0)}]$, $[\overline{W}_1^{(1)}]$, and $[\overline{W}_1^{(1)} - W_2^{(2)}]$ are the same as those given by (5.36), (5.37), and (5.38).

If the line load is parallel to the other axis, more work is entailed in getting the deflection, for the values of the W's are altered. In this case, let $y_1 = y_3 - \varepsilon_2$, $y_2 = y_3 + \varepsilon_2$, and $p_1 = p_3/2\varepsilon_2$, and take the limit just as before with $\varepsilon_2$ approaching zero.




One thing is to be observed; and this is that, before taking limits, the exponentials containing $\varepsilon_2$ must be expanded in a series form and those terms which contain $\varepsilon_2$ to the first power must be retained, as, when the W's are multiplied by $p_1$ which contains an $\varepsilon_2$ in the denominator, these terms will become constants. After this process is carried out, the deflection becomes

$$z = (p_3/p_1)\Gamma(x,y)[\overline{W}_1^{(0)} - W_2^{(0)}] \quad \text{for} \quad 0 \leq y \leq y_3 \, ,$$

$$z = (p_3/p_1)\Gamma(x,y)[\overline{W}_1^{(1)} - W_2^{(2)} + W_1^{(0)} - W_2^{(0)}] \text{ for } y_3 \leq y \leq L, \qquad (5.60)$$

where

$$
\begin{aligned}
[W_1^{(0)} - W_2^{(0)}] = [4/(\delta_1^2 - \delta_2^2)V]e^{-3cy_3/2} \Big\{ & 6c[3c\delta_2(\sinh \delta_1(y-L) \\
& - \sinh \delta_1 y \cosh \delta_2 L + \cosh \delta_2 y \sinh \delta_1 L) \\
& + 3c\delta_1(\sinh \delta_2(y - L) + \cosh \delta_1 y \sinh \delta_2 L \\
& - \sinh \delta_2 y \cosh \delta_1 L) + (\delta_1^2 - \delta_2^2)(\sinh \delta_1 y \sinh \delta_2 L \\
& - \sinh \delta_2 y \sinh \delta_1 L)][\cosh (L - y_3)\delta_1 - \cosh (L \\
& - y_3)\delta_2] + 18c^2[\cosh \delta_1(y - L) + \cosh \delta_2(y - L) \\
& - \cosh \delta_1 y \cosh \delta_2 L - \cosh \delta_2 y \cosh \delta_1 L][\delta_2 \sinh (L \\
& - y_3)\delta_1 - \delta_1 \sinh (L - y_3)\delta_2] + \sinh (L-y_3)\delta_1 [6c(\delta_2^2 \\
& - \delta_1^2)(\sinh \delta_2(y - L) + \cosh \delta_1 y \sinh \delta_2 L \\
& - \sinh \delta_2 y \cosh \delta_1 L) + (2/\delta_1)(9c^2\delta^2 - \delta_1^4 \\
& - \delta_2^4) \sinh \delta_1 y \sinh \delta_2 L + 18c^2\delta_1 \sinh \delta_2 y \sinh \delta_1 L] \\
& + \sinh L - y_3)\delta_2[6c(\delta_2^2 - \delta_1^2)(\sinh \delta_1(y - L) \\
& - \sinh \delta_1 y \cosh \delta_2 L + \cosh \delta_2 y \sinh \delta_1 L) + (2/\delta_2)(\delta_1^4 \\
& + \delta_2^4 - 9c^2\delta_1^2) \sinh \delta_2 y \sinh \delta_1 L \\
& - 18c^2\delta_2 \sinh \delta_1 y \sinh \delta_2 L] \Big\},
\end{aligned}
\tag{5.61}
$$

$$
\begin{aligned}
[W_1^{(1)} - W_2^{(2)}] = [e^{-3cy_3/2}/(\delta_1^2 - \delta_2^2)] \Big\{ & (2/\delta_1) \sinh (y-y_3)\delta_1 \\
& - (2/\delta_2) \sinh (y - y_3)\delta_2 \Big\}.
\end{aligned}
\tag{5.62}
$$

<u>Point Load</u>. The deflection due to a point load $p$ at $(x_3, y_3)$ may be obtained by combining the two line load cases just mentioned. For this case, $z$ has the following values:

$$z = \frac{p}{Na} \sum_{v=1}^{\infty} \left(\sin \alpha_v x_3 \sin \alpha_v x\right) e^{3cy/2} \left[ W_1^{(0)} - W_2^{(0)} \right]$$

for $0 \leq y \leq y_3$ ,

$$(5.63)$$

$$z = \frac{p}{Na} \sum_{v=1}^{\infty} \left(\sin \alpha_v x_3 \sin \alpha_v x\right) e^{3cy/2} \left[ W_1^{(1)} - W_2^{(2)} + W_1^{(0)} - W_2^{(0)} \right]$$

for $y_3 \leq y \leq L$ .

The W's are defined by (5.61) and (5.62).

Case $\mu = 1$. In case the $\mu$ which is part of the coefficient of $\frac{\delta^2 z}{\delta x^2}$ term in the deflection equation (5.2) be set equal to one, this equation becomes

$$L_1(z) = \Delta(\Delta z) + 6c\,\frac{\delta(\Delta z)}{\delta y} + 9c^2 \Delta z = W_1/N. \tag{5.64}$$

This is the deflection equation suggested by J. Prescott* for the slab with variable thickness. Of course $\mu$ is Poisson's ratio, and could not become unity for any ordinary elastic substance. However, a study of the differential equation with $\mu = 1$ is a profitable means of studying the solution of (5.2), and also is a help in reducing the results of the last few sections to the case of the uniformly thick slab $(c = 0)$. In actuality, this case was worked out separately, and then the resultant deflection was shown to check with that obtained by letting $\mu = 1$ in equation (5.35). The general form of $z$

*Prescott, J. Applied elasticity. p. 395. Longmans, Green and Co., London. 1924.

is nearly the same as before; i.e.,

$$z = e^{-3cy/2}\Gamma(x,y)\left[W_1^{(o)} - W_2^{(o)}\right] \quad \text{for} \quad 0 \le y \le y_1 ,$$

$$z = e^{-3cy/2}\Gamma(x,y)\left[W_1^{(1)} + W_1^{(o)} - W_2^{(o)}\right] \quad \text{for} \quad y_1 \le y \le y_2 , \qquad (5.65)$$

$$z = e^{-3cy/2}\Gamma(x,y)\left[W_1^{(1)} - W_2^{(2)} + W_1^{(o)} - W_2^{(o)}\right] \quad \text{for} \quad y_2 \le y \le L ;$$

but the W's are much less complex.

$$\left[W_1^{(o)} - W_2^{(o)}\right] = \frac{1}{3c\alpha_v(9c^2 - 4\alpha_v^2)\sinh\alpha_v L}$$

$$\Bigg\{ e^{3cy}\Bigg[\left[\frac{2e^{-3cL}\sinh\alpha_v y}{\sinh\alpha_v L}\left(\cosh\alpha_v y_1 - \cosh\alpha_v y_2\right)\right.$$

$$+ \frac{\sinh\alpha_v y}{(9c^2 - \alpha_v^2)}\Bigg[9c\alpha_v\Big(e^{-3cy_1}\cosh\alpha_v(L - y_1)$$

$$- e^{-3cy_2}\cosh\alpha_v(L - y_2)\Big) - \left(9c^2 + 2\alpha_v^2\right) \qquad (5.66)$$

$$\left(e^{-3cy_1}\sinh\alpha_v(L - y_1) - e^{-3cy_2}\sinh\alpha_v(L - y_2)\right)\Bigg]$$

$$\left. - \frac{2\sinh\alpha_v(y - L)}{\sinh\alpha_v L}\Big[\cosh\alpha_v(L - y_1) - \cosh\alpha_v(L - y_2)\Big]\right]\Bigg]$$

$$- \left[\frac{3c}{\alpha_v}\sinh\alpha_v y + 2\cosh\alpha_v y\right]\Big[\cosh\alpha_v(L - y_1)$$

$$- \cosh\alpha_v(L - y_2)\Big]\Bigg\} .$$

$$
\begin{aligned}
[\bar{W}_1^{(1)} + W_1^{(o)} - W_2^{(o)}] = {} & [1/3c\alpha_v(9c^2 - 4\,\alpha_v^2)\,\sinh\alpha_v L] \\
& e^{3cy}\Big\{[2e^{-3cL}\,\sinh\alpha_v y/\sinh\alpha_v L][\cosh\alpha_v y_1 \\
& - \cosh\alpha_v y_2] + [e^{-3cy_1}/(9c^2 - \alpha_v^2)][\sinh\alpha_v(y - L) \\
& \Big(9c^2\,\sinh\alpha_v y_1 + 2\alpha_v^2\,\sinh\alpha_v y_1 + 9c\alpha_v\,\cosh\alpha_v y_1\Big) \\
& + [e^{-3cy_2}/(9c^2 - \alpha_v^2)][\sinh\alpha_v y\,\Big(9c^2\sinh\alpha_v(L-y_2) \\
& + 2\alpha_v^2\sinh\alpha_v(L - y_2) - 9c\alpha_v\,\cosh\alpha_v(L - y_2)\Big)] \\
& - [2/\sinh\alpha_v L][\sinh\alpha_v(y - L)\,\Big(\cosh\alpha_v(L - y_1) \\
& - \cosh\alpha_v(L - y_2)\Big)]\Big\} + [1/\alpha_v^2(c_v^2 - 9c^2)] \\
& + [1/3c\alpha_v^2(9c^2 - 4\alpha_v^2)\,\sinh\alpha_v L][-\,\cosh\alpha_v y_1 \\
& \Big(3c\,\sinh\alpha_v(y - L) + 2\alpha_v\,\cosh\alpha_v(y - L)\Big) \\
& + \cosh\alpha_v(L-y_2)\Big(3c\,\sinh\alpha_v y + 2\alpha_v\,\cosh\alpha_v y\Big)].
\end{aligned}
\tag{5.67}
$$

$$
\begin{aligned}
[\bar{W}_1^{(1)} + W_2^{(2)}] = {} & [1/3c\alpha_v(9c^2 - 4\alpha_v^2)(9c^2 - \alpha_v^2)] \\
& e^{3cy}\Big\{[9c^2 + 2\alpha_v^2][e^{-3cy_1}\,\sinh\alpha_v(y - y_1) \\
& - e^{-3cy_2}\,\sinh\alpha_v(y - y_2)] - 9c\alpha_v[e^{-3cy_1}\cosh\alpha_v(y \\
& - y_1) - e^{-3cy_2}\,\cosh\alpha_v(y - y_2)]\Big\} + [1/3c\alpha_v^2(9c^2 \\
& - 4\alpha_v^2)]\Big\{3c[\cosh\alpha_v(y - y_1) - \cosh\alpha_v(y - y_2)] \\
& + 2\alpha_v[\sinh\alpha_v(y - y_1) - \sinh\alpha_v(y - y_2)]\Big\}.
\end{aligned}
\tag{5.68}
$$

In the case of a line load from $y_1$ to $y_2$ at $x = x_3$, equations (5.59) become

$$
\begin{aligned}
z &= (2p_2/Na)\sum_{v=1}^{\infty} \sin\alpha_v x_3\,\sin\alpha_v x\;\; e^{3cy/2}[\bar{W}_1^{(o)} - W_2^{(o)}] \\
&\qquad \text{from } 0 \leq y \leq y_1\,, \\
z &= (2p_2/Na)\sum_{v=1}^{\infty} \sin\alpha_v x_3\,\sin\alpha_v x\; e^{3cy/2}[\bar{W}_1^{(o)} + W_1^{(o)} - W_2^{(o)}] \\
&\qquad \text{from } y_1 \leq y \leq y_2\,,
\end{aligned}
\tag{5.69}
$$

$$z = (2p_2/Na) \sum_{v=1}^{\infty} \sin \alpha_v x_3 \sin \alpha_v x e^{3cy/2} [\overline{W}_1^{(1)} - W_2^{(2)}$$

$$+ W_1^{(o)} - W_2^{(o)}] \text{ from } y_2 \leq y \leq L , \qquad (5.69)\text{ cont.}$$

where the $W$ expressions are defined in (5.66), (5.67), and (5.68).

When the line load runs from $x_1$ to $x_2$ at $y = y_3$, equations (5.60) become

$$z = (p_3/p_1 Na)\ e^{-3cy/2}\ \Gamma(x,y)[\overline{W}_1^{(o)} - W_2^{(o)}] \text{ for } 0 \leq y \leq y_3 ,$$

$$z = (p_3/p_1 Na)\ e^{-3cy/2}\ \Gamma(x,y)[\overline{W}_1^{(1)} - W_2^{(2)} + W_1^{(o)} - W_2^{(o)}] \qquad (5.70)$$

$$\text{for } y_3 \leq y \leq L ,$$

where

$$[\overline{W}_1^{(o)} - W_2^{(o)}] = \frac{e^{3cy}}{3c\alpha_v(9c^2 - 4\alpha_v^2) \sinh \alpha_v L} \Big\{ [4\alpha_v/\sinh \alpha_v L]$$

$$[- e^{-3cL} \sinh \alpha_v y \sinh \alpha_v y_3 + \sinh \alpha_v (L$$

$$- y) \sinh \alpha_v (L - y_3)] + \frac{2e^{-3cy_3} \sinh \alpha_v y}{9c^2 - \alpha_v^2} \Big[ 9c\alpha_v$$

$$\Big(\alpha_v \sinh \alpha_v (L - y_3) + 3c \cosh \alpha_v (L - y_3)\Big) \qquad (5.71)$$

$$- (9c^2 + 2\alpha_v^2)\Big(\alpha_v \cosh \alpha_v (L - y_3) + 3c \sinh \alpha_v (L$$

$$- y_3)\Big)\Big] - 2e^{-3cy} \sinh \alpha_v (L - y_3)[3c \sinh \alpha_v y$$

$$+ 2\alpha_v \cosh \alpha_v y] \Big\} ,$$

$$[W_1^{(1)} - W_2^{(2)}] = [1/3c\alpha_v(9c^2 - 4\alpha_v^2)(9c^2 - \alpha_v^2)]e^{3c(y-y_3)}$$
$$\Big\{(9c^2 + 2\alpha_v^2)[\alpha_v \cosh \alpha_v(y - y_3) + 3c \sinh \alpha_v(y$$
$$-y_3)] - 9c\alpha_v[\alpha_v \sinh \alpha_v(y - y_3) + 3\cosh \alpha_v(y \qquad (5.72)$$
$$- y_3)]\Big\} + [2/3c\alpha_v(9c^2 - 4\alpha_v^2)]\Big\{3c \sinh \alpha_v(y$$
$$- y_3) + 2\alpha_v \cosh \alpha_v(y - y_3)\Big\}.$$

Equations (5.63) for the point load become

$$z = (P/Na)\sum_{v=1}^{\infty} \sin \alpha_v x_3 \sin \alpha_v x[W_1^{(0)} - W_2^{(0)}]$$
$$\text{for } 0 \leq y \leq y_3 ,$$
$$(5.73)$$
$$z = (P/Na)\sum_{v=1}^{\infty} \sin \alpha_v x_3 \sin \alpha_v x[W_1^{(1)} - W_2^{(2)} + W_1^{(0)} - W_2^{(0)}]$$
$$\text{for } y_3 \leq y \leq L .$$

The $W$ expressions for (5.73) are given by (5.71) and (5.72).

Comparison with the Deflection of a Slab of Uniform Thickness. As a check upon the accuracy of the results of the preceeding sections, it is worth while to let the thickness become constant and compare the thus developed expressions with the well known expressions for the deflection of a slab of uniform thickness. The deflection equation for a slab of uniform thickness is

$$\Delta(\Delta z) = W_1/N \qquad (5.74)$$

Since this is independent of $\mu$, except as $\mu$ arises in $N$, it is obvious that, by letting $\mu = 1$, one could use the simpler results derived from (5.35), just as well as (5.35) itself, as the basis for the suggested comparison. As was the case in chapter four, all exponentials must be put in series expansion form before the thickness can be made constant by letting $t_2 = t_1$ or $c \to 0$. Only a few terms of each expansion need be considered, as the rest will be vacuous when $c \to 0$. One other thing to be observed is that the $N$ of the variable thickness case becomes the $N$ of the constant thickness case. In the light of these observations, equations, (5.71) and (5.72) for the point load become

$$\left[W_1^{(o)} - W_2^{(o)}\right] = \frac{(y_3 - L)\sinh\alpha_v y \cosh\alpha_v(L - y_3)}{\alpha_v^2 \sinh\alpha_v L} + \sinh\alpha_v(L - y_3)\left\{\frac{L\cosh\alpha_v L \sinh\alpha_v y}{\alpha_v^2 \sinh^2\alpha_v L} - \frac{y\cosh\alpha_v y}{\alpha_v^2 \sinh\alpha_v L} + \frac{\sinh\alpha_v y}{\alpha_v^3 \sinh\alpha_v L}\right\}, \tag{5.75}$$

$$\left[W_1^{(1)} - W_2^{(2)}\right] = \frac{(y - y_3)\cosh\alpha_v(y - y_3)}{\alpha_v^2} - \frac{\sinh\alpha_v(y - y_3)}{\alpha_v^3}. \tag{5.76}$$

The deflection (5.73) becomes

$$z = (P/Na)\sum_{v=1}^{\infty} \sin\alpha_v x_3 \sin\alpha_v x\left[W_1^{(o)} - W_2^{(o)}\right] \quad \text{from } 0 \le y \le y_3,$$

$$z = (P/Na)\sum_{v=1}^{\infty} \sin\alpha_v x_3 \sin\alpha_v x\left[W_1^{(1)} - W_2^{(2)} + W_1^{(o)} - W_2^{(o)}\right] \quad \text{for } y_3 \le y \le L,$$

where the W's are defined by (5.75) and (5.76). This agrees exactly with the value obtained by H. Schmidt.*

In the case of a uniform load covering the whole slab

$$z = \frac{p_1}{Na} \sum_{v=1,3,5,7\ldots} \frac{1}{\alpha_v} \sin \alpha_v x \left\{ 2 \sinh \alpha_v y \left[ y + \frac{L(1 - \cosh \alpha_v L)}{\sinh^2 \alpha_v L} + \frac{2}{\alpha_v}\left(\frac{\cosh \alpha_v L - 1}{\sinh \alpha_v L}\right)\right] + \cosh \alpha_v y \left[ \frac{2y(1 - \cosh \alpha_v L)}{\sinh \alpha_v L} - \frac{4}{\alpha_v}\right] + \frac{4}{\alpha_v} \right\} .$$

Although different in form, this last expression is equivalent to those given by Schmidt* and Nadai** for plates under a uniform load. The same process can be shown to check the various other results of this paper.

*Schmidt, H. Biegung der frei aufliegender Rechteckplatte mit statischer, rechteckig berandeter Lastverteilung. Zeit. f. Phys., 68:423-432. 1931.

**Nadai, A. Elastische Platten. p. 123. J. Springer, Berlin.

## VI. CONCLUSION.

1. The partial differential deflection equation for a slab having a variable thickness and a cylindrical central surface of large radius is worked out in terms of thickness and curvature.

2. A thickness function, exponential in form, is found to simplify the solution of the deflection equation and produce a slab which approximates the modern paving slab. This function is used to define all the slabs considered in the remainder of the paper.

3. The deflections, strains, and stresses for a uniformly loaded, infinitely long, curved slab (width L) are developed for two cases:

a. Edges simply supported

b. Edges clamped.

4. The deflections, etc., indicated in 3 are shown to be similar to the deflections, etc., of a uniformly loaded beam (unit width) whose cross-sectional variation is defined by the thickness function mentioned in 2.

5. The curvature of the simply supported slab of 3 is shown to have such a small effect upon the deflections, etc., that this curvature is neglected in studying the simply supported, rectangular slab of 6.

6. The deflection expressions for a simply supported, rectangular slab (length a, width L) under rectangular, line, and point loads are worked out by two methods:

a. Integral method

b. Fourier series method.

The results obtained by these two methods are in complete agreement. The deflections are developed in the form of an infinite series of trigonometric and hyperbolic terms.

7. Special cases of these expressions for the deflection are modified by letting the thickness of the slab become constant, and the resultant expressions are shown to be in agreement with the known deflection expressions for plates of uniform thickness.